Sample Solutions Exercise 2

Exercise 2.1: Simple Lie algebras and the exponential function (6P)

Consider an abstract operator λ obeying $\lambda^2 = -\mathbb{1}$, where $\mathbb{1} = \lambda^0$ is the identity.

(a) Write down the Taylor series of $\Lambda = \exp(\phi\lambda)$, where $\phi \in \mathbb{R}$. (1P)

(b) Separate the Taylor series into an even and an odd part in order to show that (2P)

$$\exp(\phi\lambda) = \mathbb{1}\cos\phi + \lambda\sin\phi.$$

(c) Apply the result from (b) to the representations $\lambda = i$ and $\lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (1P)

(d) Repeat (b) for $\lambda^2 = +\mathbb{1}$ and find a non-trivial 2×2 matrix representation. (2P)

Sample Solution

(a) The Taylor series reads (1P)

$$\Lambda = \exp(\phi\lambda) = \sum_{k=0}^{\infty} \frac{\phi^k \lambda^k}{k!}.$$

(b) The Taylor series can be separated into an even and an odd part (2P)

$$\begin{aligned} \exp(\phi\lambda) &= \sum_{n=0}^{\infty} \frac{\phi^{2n}\lambda^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\phi^{2n+1}\lambda^{2n+1}}{(2n+1)!} \\ &= \mathbb{1}\sum_{n=0}^{\infty} \frac{\phi^{2n}(-1)^n}{(2n)!} + \lambda\sum_{n=0}^{\infty} \frac{\phi^{2n+1}(-1)^n}{(2n+1)!} = \mathbb{1}\cos\phi + \lambda\sin\phi. \end{aligned}$$

(c) For the complex representation $\lambda = i$ we get the well-known formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

which is a phase factor describing a rotation in the complex plane. For the matrix representation we get:

$$\exp\left[\phi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\phi + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sin\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

which is just a rotation matrix in $\mathbb{R}^2$. (1P)

(d) For $\lambda^2 = +\mathbb{1}$ we only have to delete the alternating signs $(-1)^n$, i.e.: (1P)

$$\exp(\phi\lambda) = \mathbb{1}\sum_{n=0}^{\infty} \frac{\phi^{2n}}{(2n)!} + \lambda\sum_{n=0}^{\infty} \frac{\phi^{2n+1}}{(2n+1)!} = \mathbb{1}\cosh\phi + \lambda\sinh\phi.$$

A possible matrix representation is (1P)

$$\exp\left[\phi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh\phi + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sinh\phi = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

In principle we could use any of the three Pauli matrices.

Exercise 2.2: Isometries **(6P)**

An isometry Λ is a transformation that does not change the metric tensor, i.e.

$$g^{ij} = \Lambda^i{}_k \Lambda^j{}_l g^{kl}. \tag{*}$$

Let us assume that $\Lambda = \exp(\epsilon\lambda)$ is an infinitesimal transformation with $\epsilon \ll 1$.

(a) Write the series expansion of $\Lambda = \exp(\epsilon\lambda)$ in components to first order in ϵ. (1P)

(b) Insert (a) into (*), drop all terms with ϵ^2, and derive a condition for λ^{ij}. (2P)

(c) Show that in a 2-dimensional space with a given metric g_{ij}, the matrix $\lambda^i{}_k$ is proportional to (1P)

$$\begin{pmatrix} \lambda^1{}_1 & \lambda^1{}_2 \\ \lambda^2{}_1 & \lambda^2{}_2 \end{pmatrix} \propto \begin{pmatrix} g_{12} & g_{22} \\ -g_{11} & -g_{12} \end{pmatrix}$$

(d) Use the results of the previous exercise to compute the full (non-infinitesimal) isometry $\Lambda(\phi) = \exp(\phi\lambda)$ for the special cases (2P)

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Sample Solution

(a) The series expansion of $\Lambda = \exp(\epsilon\lambda)$ to first order in ϵ reads (1P)

$$\Lambda = \mathbb{1} + \epsilon\lambda + \mathcal{O}(\lambda^2) \quad \Rightarrow \quad \Lambda^i{}_k = \delta^i_k + \epsilon\lambda^i{}_k + \mathcal{O}(\epsilon^2).$$

(b) If we insert (a) into (*), dropping all ϵ^2-terms, we get: (1P)

$$g^{ij} = \left(\delta^i_k\delta^j_l + \epsilon\lambda^i{}_k\delta^j_l + \epsilon\delta^i_k\lambda^j{}_l\right) g^{kl} = g^{ij} + \epsilon\lambda^i{}_k g^{kj} + \epsilon\lambda^j{}_l g^{il}$$

$$\Rightarrow \quad \epsilon\lambda^i{}_k g^{kj} + \epsilon\lambda^j{}_l g^{il} = 0$$

$$\Rightarrow \quad \lambda^{ij} + \lambda^{ji} = 0.$$

This means that λ^{ij} has to be antisymmetric. (1P)

(c) In 2d the antisymmetry implies that $\lambda^{21} = -\lambda^{12}$ and $\lambda^{11} = \lambda^{22} = 0$. Using the given 2×2 metric in order to lower the second index in $\lambda^i{}_j = \lambda^{ik} g_{kj}$, we arrive at

$$\lambda^1{}_1 = \lambda^{12} g_{21}\,, \quad \lambda^1{}_2 = \lambda^{12} g_{22}\,, \quad \lambda^2{}_1 = -\lambda^{12} g_{11}\,, \quad \lambda^2{}_2 = -\lambda^{12} g_{12}.$$

Here λ_{12} plays the role of a proportionality factor: (1P)

$$\lambda^i{}_j = \begin{pmatrix} \lambda^1{}_1 & \lambda^1{}_2 \\ \lambda^2{}_1 & \lambda^2{}_2 \end{pmatrix} = \lambda^{12} \begin{pmatrix} g_{12} & g_{22} \\ -g_{11} & -g_{12} \end{pmatrix}$$

(d)
- $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \lambda^i{}_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Rightarrow \quad \Lambda^i{}_j = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$

- $g_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \lambda^i{}_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \Lambda^i{}_j = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}.$
  (2P)

(Σ = 12P)