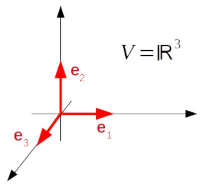
Sample Solutions Exercise 1

Exercise 1.1: Skew coordinates (6P)

Consider the Euclidean vector space $V = \mathbb{R}^3$ equipped with the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the standard scalar product $g(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}$. Define the vectors



$$\begin{aligned} \tilde{\mathbf{e}}_1 &= \mathbf{e}_1 - \mathbf{e}_2 \\ \tilde{\mathbf{e}}_2 &= \mathbf{e}_1 + \mathbf{e}_2 \\ \tilde{\mathbf{e}}_3 &= \mathbf{e}_1 - 2\mathbf{e}_2 - \mathbf{e}_3 \end{aligned}$$

(a) Show that $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ is a basis of V. (1P)

(b) Let $\mathbf{u} = u^i \mathbf{e}_i = \tilde{u}^i \tilde{\mathbf{e}}_i$ be a given vector. Express the new components $\tilde{u}^i$ explicitly in terms of the old components u_i. (1P)

(c) Find the metric tensor $\tilde{g}_{ij}$ in the basis $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$. (1P)

(d) Compute the corresponding metric tensor $\tilde{g}^{ij}$ in the dual space. (1P)

(e) Let $\alpha = \mathbf{e}^1 - \mathbf{e}^3$ be a given linear form. Derive an equation for the hyperplane where this form vanishes, that is, find a relation among the components u^i of a vector $\mathbf{u} = u^i \mathbf{e}_i$ in such a way that $\alpha(\mathbf{u}) = 0$. (1P)

(f) Find the representation $\alpha = \tilde{\alpha}_j \tilde{\mathbf{e}}^j$ in the dual basis $\{\tilde{\mathbf{e}}^1, \tilde{\mathbf{e}}^2, \tilde{\mathbf{e}}^3\}$. (1P)

Sample Solution

(a) To this end we have to show that $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3$ are linearly independent. This can be done in various ways, for example by writing

$$\begin{pmatrix} \tilde{\mathbf{e}}_1 & \tilde{\mathbf{e}}_2 & \tilde{\mathbf{e}}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

and checking that the determinant of this matrix does not vanish. Another way would be to argue that $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$ point in different direction, thus they are linearly independent, and that $\tilde{\mathbf{e}}_3$, which involves the contribution $-\mathbf{e}_3$, is also linearly independent of $\tilde{\mathbf{e}}_1$ and $\tilde{\mathbf{e}}_2$. (1P)

(b) According to the lecture notes the basis vectors are related by

$$\mathbf{e}_i = \tilde{\mathbf{e}}_j A^j{}_i \quad \text{and} \quad \tilde{\mathbf{e}}_j = \mathbf{e}_k B^k{}_j$$

while the components are related by

$$\tilde{u}^j = A^j{}_i u^i \quad \text{and} \quad u^k = B^k{}_j \tilde{u}^j$$

where the transformation matrices are given by

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{A} = \mathbf{B}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 3 \\ 1 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

From this we read off:

$$\tilde{u}^1 = \tfrac{1}{2}u^1 - \tfrac{1}{2}u^2 + \tfrac{3}{2}u^3 \qquad \tilde{u}^2 = \tfrac{1}{2}u^1 + \tfrac{1}{2}u^2 - \tfrac{1}{2}u^3 \qquad \tilde{u}^3 = -u^3.$$

(c) The metric tensor transforms like a form in each index (see lecture notes), that is:

$$\tilde{g}_{ij} = \underbrace{g_{kl}}_{=\delta_{kl}} B^k{}_i B^l{}_j = (\mathbf{B}^T\mathbf{B})_{ij} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & -1 \\ 3 & -1 & 6 \end{pmatrix}_{ij}$$

Note: The matrix is symmetric and the eigenvalues are positive, hence the metric is still positive definite like the original one. In fact, a basis transformation cannot change the signature of a metric. The eigenvalues are allowed to change because the transformation is not an isometry.

(d) According to the lecture notes, the metric tensor $\tilde{g}^{ij}$ in the covector space is the inverse of $\tilde{g}_{ij}$:

$$\tilde{g}^{ij} = \frac{1}{4}\begin{pmatrix} 11 & -3 & -6 \\ -3 & 3 & 2 \\ -6 & 2 & 4 \end{pmatrix}^{ij}$$

(e) With the representation $\mathbf{u} = u^i\mathbf{e}_i$ and the universal relation $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$ we get

$$\alpha(\mathbf{u}) = \mathbf{e}^1(\mathbf{u}) - \mathbf{e}^3(\mathbf{u}) = u^1 - u^3.$$

The hyperplane $\alpha(\mathbf{u}) = 0$ is therefore described by the linear relation $u^1 = u^3$.

(f) In the original basis we have $\alpha_1 = 1, \alpha_2 = 0$, and $\alpha_3 = -1$. According to the lecture notes, these components transform as

$$\tilde{\alpha}_l = \alpha_k B^k{}_l = B^1{}_l - B^3{}_l\,,$$

giving

$$\tilde{\alpha}_1 = 1\,, \quad \tilde{\alpha}_2 = 1\,, \quad \tilde{\alpha}_3 = 2\,.$$

Hence

$$\alpha = \tilde{\mathbf{e}}^1 + \tilde{\mathbf{e}}^2 + 2\tilde{\mathbf{e}}^3.$$

$(\Sigma = 6\mathbf{P})$