Exercise 4.1: Lorentz transform in general boost direction (4P)

Consider a rest frame S and an inertial frame S' moving relative to S with velocity $\vec{v}$.

(a) Starting from the usual Lorentz transformation in x-direction (see lecture notes), show that the 4×4 matrix of the Lorentz transformation in arbitrary direction is given by (2P)

$$\Lambda = \begin{pmatrix} \gamma & \frac{\gamma}{c}\vec{v}^T \\ \frac{\gamma}{c}\vec{v} & \mathbb{1}_{3\times 3} + \frac{\gamma-1}{v^2}\vec{v}\vec{v}^T \end{pmatrix}$$

where $\vec{v}^T$ is the transposed velocity vector and $\vec{v}^T\vec{v}$ the 3×3 dyadic product.

(b) As we will see in forthcoming lectures, the tensor of the electromagnetic field is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

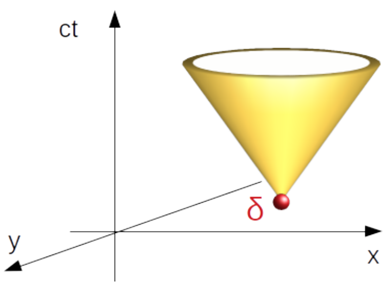
Suppose that the resting observer in S sees the fields $\vec{E}$ and $\vec{B}$. Compute the fields $\vec{E}'$ and $\vec{B}'$ seen from an observer S' moving in x-direction at velocity v. (2P)
Hint: One possible way would be to transform the electromagnetic field tensor $F^{\mu\nu}$ by the Lorentz transformation and to identify its components.

Exercise 4.2: Causality in the wave equation (8P)

Let us consider the wave equation of a scalar field $\Phi(\mathbf{x})$ with an external source term $J(\mathbf{x})$, described by the system of inhomogeneous linear partial differential equations

$$\Box\Phi(\mathbf{x}) = J(\mathbf{x}).$$

The purpose of this exercise is to demonstrate causality, meaning that a local perturbation caused by $J(\mathbf{x})$ can only spread within its own lightcone.



(a) Prove that the solution of the inhomogeneous wave equation can be written as

$$\Phi(\mathbf{x}) = \int \mathrm{d}^4 y\, G(\mathbf{x}-\mathbf{y}) J(\mathbf{y})\,,$$

where the Green function $G(\mathbf{x})$ is a solution of $\Box G(\mathbf{x}) = \delta^4(\mathbf{x})$. (1P)

(b) Show that $\delta^4(\mathbf{x}) = \frac{1}{16\pi^4}\int d^4k\, e^{-ik^\mu x_\mu}$. (1P)

(c) Use (a) and (b) to prove that $G(\mathbf{x}) = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4k\, \frac{e^{-ik^\mu x_\mu}}{k^\nu k_\nu}$. (1P)

(d) Evaluate the integral over k_0 by replacing $k_0 \to k_0 - i\epsilon$, computing the integral in the complex plane, and then take $\epsilon \to 0$. (This choice corresponds to selecting the retarded Greens function). (2P)

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(e) Use spherical coordinates for the integration over d^3k in order to show that (1P)

$$G(\mathbf{x}) = -\frac{\theta(x^0)}{(2\pi)^2}\frac{2}{r}\int_0^\infty \mathrm{d}k\,\sin(kx^0)\sin(kr),$$

where $\theta(x^0)$ denotes the Heaviside step function, selecting $x_0 > 0$.

(f) Compute the integral given in (e). (1P)

(g) Show that your result obtained in (f) can also be written in the form (1P)

$$G(\mathbf{x}) = -\frac{\theta(x^0)}{2\pi}\delta(x^\mu x_\mu).$$

Hint: For this you need to evaluate Dirac-delta of a function $\delta(f(x))$.

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, May 27 at 12:00.