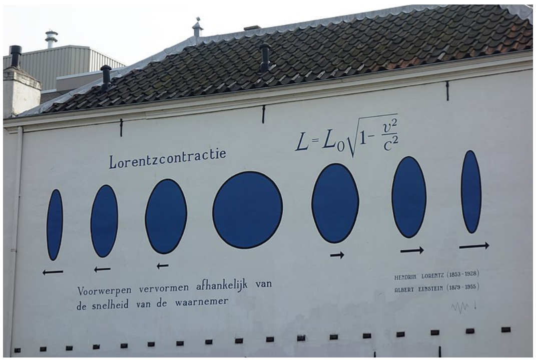
Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2022



[Lorentz contraction explained (Wikimedia)]

Exercise 3.1: Fourier transformation (6P)

The Fourier transformation F maps a function $f : \mathbb{C} \to \mathbb{C}$ to a different function $\tilde{f} : \mathbb{C} \to \mathbb{C}$ by the following linear transformation:

$$F : f \mapsto \tilde{f} : \quad \tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iyx} f(x)\, \mathrm{d}x.$$

We assume that the functions are chosen in such a way that their Fourier transform exists.

(a) Show that $F^2 = F \circ F \neq \mathbb{1}$ and that $F^4 = \mathbb{1}$. What is the meaning of F^2? (2P)
You can use the relation $\int_{-\infty}^{+\infty} e^{ix}\, \mathrm{d}x = 2\pi\delta(x)$.

(b) Consider a Gaussian bell curve $f(x) = e^{-ax^2}$. Determine a in such a way that f is an eigenfunction of F. Please use quadratic completion. (2P)

(c) The scalar product of two functions can be defined as

$$g(f, h) = \int_{-\infty}^{+\infty} f^*(x) h(x)\, \mathrm{d}x\,.$$

Show that for this metric the Fourier transformation is an isometry, i.e.,: (2P)

$$g(f, h) = g(\tilde{f}, \tilde{h})\,.$$

Exercise 3.2: Lorentz transformations in 2+1 dimensions (6P)

In a 2+1 dimensional Minkowski space let us consider Lorentz transformations $\Lambda_x(\theta)$ in x-direction , Lorentz transformations $\Lambda_y(\theta)$ in y-direction, as well as ordinary rotations $R(\phi)$ in the xy-plane. These corresponding transformation matrices are given by

$$\Lambda_x(\beta) = \begin{pmatrix} \cosh\theta & \sinh\theta & \\ \sinh\theta & \cosh\theta & \\ & & 1 \end{pmatrix}, \quad \Lambda_y(\beta) = \begin{pmatrix} \cosh\theta & & \sinh\theta \\ & 1 & \\ \sinh\theta & & \cosh\theta \end{pmatrix}, \quad R(\phi) = \begin{pmatrix} 1 & & \\ & \cos\phi & -\sin\phi \\ & \sin\phi & \cos\phi \end{pmatrix}$$

where $\theta = \mathrm{artanh}(\beta)$ is the rapidity and $\beta = v/c$. The aim of this exercise is to compute the Lorentz transformation in arbitrary direction with velocity $\vec{v} = (v_x, v_y)^T = (v_1, v_2)^T$.

Please turn over ⇒

(a) Compute β and ϕ in such a way that $\vec{\beta} = \vec{v}/c = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ (1P)

(b) Prove that a Lorentz transformation in direction $\vec{\beta}$ is given by the 3×3 matrix

$$\Lambda(\vec{\beta}) = \left(\begin{array}{c|c} \gamma & \gamma\vec{\beta}^T \\ \hline \gamma\vec{\beta} & \mathbb{1}_{2\times 2} + \mathbf{B} \end{array}\right)$$

where $\mathbf{B}$ is a 2×2 matrix given by the dyadic product $\mathbf{B} = \frac{\gamma-1}{\beta^2}\mathbf{b}^T \otimes \mathbf{b}$ with the components $B_{ij} = \frac{\gamma-1}{\beta^2}\beta_i\beta_j$. (3P)

Hint: Find an argument explaining why $\Lambda(\vec{\beta}) = R(\phi)\Lambda_x(\beta)R(-\phi)$ with the angle ϕ and β computed in (a) and calculate the expression on the right hand side.

(c) Is it possible to find a direction $\vec{\delta}$ such that two subsequent Lorentz boosts by $\Lambda_x(\beta)$ and $\Lambda_y(\beta)$ can be expressed as a single boost $\Lambda(\vec{\delta})$? (2P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, May 20 at 12:00.