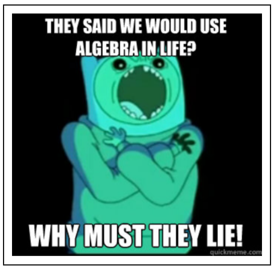
Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2022



Lie algebras [quickmeme.com]

Exercise 2.1: Simple Lie algebras and the exponential function (6P)

Consider an abstract operator λ obeying $\lambda^2 = -\mathbb{1}$, where $\mathbb{1} = \lambda^0$ is the identity.

(a) Write down the Taylor series of $\Lambda = \exp(\phi\lambda)$, where $\phi \in \mathbb{R}$. (1P)

(b) Separate the Taylor series into an even and an odd part in order to show that (2P)

$$\exp(\phi\lambda) = \mathbb{1}\cos\phi + \lambda\sin\phi.$$

(c) Apply the result from (b) to the representations $\lambda = i$ and $\lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (1P)

(d) Repeat (b) for $\lambda^2 = +\mathbb{1}$ and find a non-trivial 2×2 matrix representation. (2P)

Exercise 2.2: Isometries (6P)

An isometry Λ is a transformation that does not change the metric tensor, i.e.

$$g^{ij} = \Lambda^i{}_k \Lambda^j{}_l g^{kl}. \qquad (*)$$

Let us assume that $\Lambda = \exp(\epsilon\lambda)$ is an infinitesimal transformation with $\epsilon \ll 1$.

(a) Write the series expansion of $\Lambda = \exp(\epsilon\lambda)$ in components to first order in ϵ. (1P)

(b) Insert (a) into (*), drop all terms with ϵ^2, and derive a condition for λ^{ij}. (2P)

(c) Show that in a 2-dimensional space with a given metric g_{ij}, the matrix $\lambda^i{}_k$ is proportional to (1P)

$$\begin{pmatrix} \lambda^1{}_1 & \lambda^1{}_2 \\ \lambda^2{}_1 & \lambda^2{}_2 \end{pmatrix} \propto \begin{pmatrix} g_{12} & g_{22} \\ -g_{11} & -g_{12} \end{pmatrix}$$

(d) Use the results of the previous exercise to compute the full (non-infinitesimal) isometry $\Lambda(\phi) = \exp(\phi\lambda)$ for the special cases (2P)

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Σ = **12P**)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, May 13 at 12:00.