Sample Solutions Exercise 10

Exercise 10.1: Helicity (4P)

The *helicity operator* is an operator (4×4-matrix) acting in spinor space. In the Dirac representation it is defined as the projection of the spin onto the direction of momentum

$$h(\vec{p}) = \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{p}\cdot\vec{\sigma} & \\ & \vec{p}\cdot\vec{\sigma} \end{pmatrix},$$

where $\vec{p}$ is the 3-momenteum and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T = (\sigma^x, \sigma^y, \sigma^z)^T$ is the vector operator consisting of Pauli matrices.

(a) Verify that $\big(h(\mathbf{p})\big)^2 = \mathbb{1}$. (1P)

(b) Show that for given momentum $\vec{p}$, the operators $P_\pm^{hel.}(\vec{p}) = \frac{1}{2}(\mathbb{1} \pm h(\vec{p}))$ is a complete set of orthogonal projection operators. (2P)

(c) Read about helicity and chirality in textbooks and explain qualitatively their difference. (1P)

Sample Solution

(a) This relation can be shown by using the Pauli algebra $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_2$:

$$\big(h(\vec{p})\big)^2 = \frac{1}{|\vec{p}|^2}\begin{pmatrix} p^i\sigma_i & \\ & p^i\sigma_i \end{pmatrix}^2 = \frac{1}{\vec{p}\cdot\vec{p}}\begin{pmatrix} p^ip^j\sigma_i\sigma_j & \\ & p^ip^j\sigma_i\sigma_j \end{pmatrix},.$$

$$= \frac{1}{2\vec{p}\cdot\vec{p}}\begin{pmatrix} p^ip^j\{\sigma_i,\sigma_j\} & \\ & p^ip^j\{\sigma_i,\sigma_j\} \end{pmatrix} = \frac{1}{2\vec{p}\cdot\vec{p}}\begin{pmatrix} \vec{p}\cdot\vec{p} & \\ & \vec{p}\cdot\vec{p} \end{pmatrix} = \mathbb{1}_4$$

(b) We have to show that (2P)

- they are projection operators:

$$(P_\pm)^2 = \frac{1}{4}(\mathbb{1} \pm h)^2 = \frac{1}{4}(\mathbb{1} \pm 2h + h^2) = \frac{1}{4}(\mathbb{1} \pm 2h + \mathbb{1}) = P_\pm.$$

- they are orthogonal:

$$P_+P_- = \frac{1}{4}(\mathbb{1}+h)(\mathbb{1}-h) = \frac{1}{4}(\mathbb{1}-h^2) = \frac{1}{4}(\mathbb{1}-\mathbb{1}) = 0.$$

- they are complete (span the full spinor space):

$$P_+ + P_= \frac{1}{2}(\mathbb{1}+h) + \frac{1}{2}(\mathbb{1}-h) = \mathbb{1}.$$

(c) The helicity of a particle is right-handed if the direction of its spin is the same as the direction of its motion. It is left-handed if the directions of spin and motion are opposite. The chirality of a particle is more abstract: It is determined by whether the particle transforms in a right- or left-handed representation of the Poincaré group. (1P)

Exercise 10.2: Covariance of the Dirac equation (8P)

According to the Lecture Notes, the 4-component Dirac wave function $\psi(\mathbf{x})$ transforms under $SO^+(3,1)$-transformations in the $(\alpha\beta)$-plane by the 'angle' $\theta_{(\alpha\beta)} \in \mathbb{R}$ as

$$\psi \to S\big(\theta_{(\alpha\beta)}\big)\psi \quad \text{where} \quad S\big(\theta_{(\alpha\beta)}\big) = \exp\Big(\frac{i}{2}\theta_{(\alpha\beta)}\sigma_{(\alpha\beta)}\Big) \quad \text{and} \quad \sigma_{(\alpha\beta)} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta].$$

(a) Convince yourself that in a representation of your choice the adjoint (= hermitean conjugate) of the γ-matrices is given by (1P)

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad \text{and} \quad {\gamma_\mu}^\dagger = \gamma_0\gamma_\mu\gamma_0$$

(b) Use (a) to show algebraically that the rotation generators $\sigma_{(12)}, \sigma_{(13)}, \sigma_{(23)}$ are hermitean while the boost generators $\sigma_{(01)}, \sigma_{(02)}, \sigma_{(03)}$ are anti-hermitean. (2P)

(c) Show that

$$\psi^\dagger\psi \;=\; \big(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*\big)\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \;=\; \sum_{k=1}^{4}\psi_k^*\psi_k$$

is **not** invariant under Lorentz boosts. (1P)

(d) Prove that $S^\dagger_{(\alpha\beta)} = \gamma_0\, S^{-1}_{(\alpha\beta)}\, \gamma_0$. *Hint:* Check that $\sigma^\dagger_{(\alpha\beta)} = \gamma_0\, \sigma_{(\alpha\beta)}\, \gamma_0$. (2P)

(e) Show that instead $\bar\psi\psi$ is invariant under Lorentz boosts as well as under rotations, where $\bar\psi := \psi^\dagger\gamma^0$ is the so-called *adjoint spinor*. (1P)

(f) Show that $\bar\psi\gamma^\nu\psi$ transforms covariantly as a 4-vector.
Hint: The easiest way is to show that this condition is equivalent to $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\gamma^\nu$ which we have already proven in the Lecture Notes. (1P)

Sample Solution

(a) This is trivial to check in a given representation of the γ-matrices. Proof works well in all representations given the lecture notes.

(b) The generators are given by

$$\sigma_{(\alpha\beta)} \;=\; +\frac{i}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha).$$

We calculate the Hermitean conjugate of these matrices using the result of (a):

$$\sigma^\dagger_{(\alpha\beta)} = -\frac{i}{2}(\gamma^\dagger_\beta\gamma^\dagger_\alpha - \gamma^\dagger_\alpha\gamma^\dagger_\beta) = +\frac{i}{2}(\gamma_0\gamma_\alpha \underbrace{\gamma_0\gamma_0}_{=\mathbb{1}} \gamma_\beta\gamma_0 - \gamma_0\gamma_\beta \underbrace{\gamma_0\gamma_0}_{=\mathbb{1}} \gamma_\alpha\gamma_0)$$

$$\Rightarrow \quad \sigma^\dagger_{(\alpha\beta)} = +\frac{i}{2}(\gamma_0\gamma_\alpha\gamma_\beta\gamma_0 - \gamma_0\gamma_\beta\gamma_\alpha\gamma_0) = \gamma_0\sigma_{(\alpha\beta)}\gamma_0$$

Now we have to distinguish two cases. In the case of Lorentz boosts, where $\alpha = 0$ and $\beta = 1, 2, 3$ we obtain

$$\sigma^\dagger_{(0\beta)} = +\frac{i}{2}[\gamma_\beta, \gamma_0] = -\frac{i}{2}[\gamma_0, \gamma_\beta] = -\sigma_{(0\beta)}$$

where we used $\gamma_0\gamma_0 = \mathbb{1}$, hands the booster generators are anti-hermitean. Contrarily, for ordinary rotations, where $\alpha \neq 0$ and $\beta \neq 0$, we can anti-commute γ_α and γ_β with the γ_0's, giving

$$\sigma^\dagger_{(\alpha\beta)} = +\frac{i}{2}(\gamma_\alpha\gamma_0\gamma_0\gamma_\beta - \gamma_\beta\gamma_0\gamma_0\gamma_\alpha) = +\sigma_{(\alpha\beta)}$$

proving that the generators corresponding to rotations are Hermitean.

(c) Under a $SO^+(3.1)$-transformation the spinor were function transforms as follows:

$$\psi \to S\psi\,, \quad \psi^\dagger \to \psi^\dagger S^\dagger.$$

Hence the term in question transforms as:

$$\psi^\dagger\psi \to \psi^\dagger S^\dagger S\psi.$$

Invariance requires that $S^\dagger S = \mathbb{1}$, meaning that S has to be unitary. Since $S = \exp(\frac{i}{2}\theta\sigma)$, this means in turn that the corresponding generator has to be Hermitean. But in (b) we showed that this is only true for ordinary rotations, but not for Lorentz boosts.

(d) Because of $\sigma^\dagger_{(\alpha\beta)} = \gamma_0\,\sigma_{(\alpha\beta)}\,\gamma_0$ and $\gamma_0\gamma_0 = \mathbb{1}$ we can conclude that any power transforms like

$$\left[\sigma^\dagger_{(\alpha\beta)}\right]^n = \gamma_0\left[\sigma_{(\alpha\beta)}\right]^n\gamma_0$$

The transformation matrix S is the exponential of σ and therefore a power series. It follows that

$$\begin{aligned} S^\dagger\big(\theta_{(\alpha\beta)}\big) &= \left[\exp\Big(\frac{i}{2}\theta_{(\alpha\beta)}\sigma_{(\alpha\beta)}\Big)\right]^\dagger = \exp\Big(-\frac{i}{2}\theta_{(\alpha\beta)}\sigma^\dagger_{(\alpha\beta)}\Big) \\ &= \sum a_k\big[\sigma^\dagger_{(\alpha\beta)}\big]^k = \sum a_k\gamma_0\left[\sigma_{(\alpha\beta)}\right]^k\gamma_0 \\ &= \gamma_0\exp\Big(-\frac{i}{2}\theta_{(\alpha\beta)}\sigma_{(\alpha\beta)}\Big)\gamma_0 = \gamma_0\,S^{-1}\big(\theta_{(\alpha\beta)}\big)\,\gamma_0 \end{aligned}$$

(e) Suppressing the argument in $\psi(\mathbf{x})$ and $S\big(\theta_{(\alpha\beta)}\big)$ this can be proven as follows:

$$\begin{aligned} \bar\psi\psi = \psi^\dagger\gamma^0\psi \quad \to \quad \psi^\dagger S^\dagger\gamma^0 S\psi &= -\psi^\dagger S^\dagger\gamma_0 S\psi \\ &= -\psi^\dagger\gamma_0 S^{-1}\underbrace{\gamma_0\gamma_0}_{=\mathbb{1}}S\psi \\ &= +\psi^\dagger\gamma^0\underbrace{S^{-1}S}_{=\mathbb{1}}\psi = \bar\psi\psi \qquad \square \end{aligned}$$

(f) On the one hand we know that this expression transforms as

$$\begin{aligned}\bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi \quad \rightarrow \quad & \psi^\dagger S^\dagger\gamma^0\gamma^\mu S\psi \;=\; -\psi^\dagger S^\dagger\gamma_0 S\psi \\ &= \; -\psi^\dagger\gamma_0 S^{-1}\underbrace{\gamma_0\gamma_0}_{=\mathbb{1}}\gamma^\mu S\psi \\ &= \; +\psi^\dagger\gamma^0\, S^{-1}\gamma^\mu S\,\psi\end{aligned}$$

This transformation is that of a 4-vector if

$$\bar{\psi}\gamma^\mu\psi \quad \rightarrow \quad \Lambda^\mu{}_\nu\,\bar{\psi}\gamma^\nu\psi \;=\; \psi^\dagger\gamma^0\,\Lambda^\mu{}_\nu\gamma^\nu\,\psi$$

This is just the last formula in the paragraph *Covariant spinor transformation* in the Lecture Notes, namely

$$S^{-1}\gamma^\mu S \;=\; \Lambda^\mu{}_\nu\gamma^\nu.$$

(Σ = 12P)