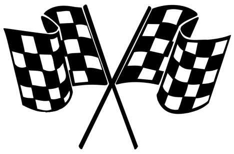
Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2021



Am Ziel!

Exercise 12.1: Charge conjugation operator (6P)

Under a Lorentz transformation $\mathbf{x} \to \Lambda\mathbf{x}$ with $\Lambda = \exp\big(\sum_{\alpha,\beta} \theta^{(\alpha\beta)} \lambda_{(\alpha\beta)}\big)$ the spinors change as $\psi(\mathbf{x}) \to \mathbf{S}\psi(\mathbf{x})$ with $\mathbf{S} = \exp\big(\frac{i}{2}\sum_{\alpha,\beta} \theta^{(\alpha\beta)} \sigma_{(\alpha\beta)}\big)$. In this exercise let us consider the charge conjugation operator $\mathbf{C}$ which is defined as the unitary matrix in spinor space taking[1]

$$\gamma^\mu \to -(\gamma^\mu)^* = \mathbf{C}^\dagger \gamma^\mu \mathbf{C} \qquad \text{and} \qquad \gamma_\mu \to -(\gamma_\mu)^* = \mathbf{C}^\dagger \gamma_\mu \mathbf{C}\,.$$

(a) Prove that if ψ is a solution of the Dirac equation $(i\gamma^\mu\partial_\mu - M)\psi = 0$, then the charge-conjugate spinor $\psi^{(c)} := \mathbf{C}\psi^*$ is a solution as well. (2P)

(b) Calculate $\mathbf{C}^\dagger \sigma_{(\alpha\beta)} \mathbf{C}$. (1P)

(c) Show that $\mathbf{C}^\dagger \mathbf{S} \mathbf{C} = \mathbf{S}^*$. (1P)

(d) Show that $\psi^{(c)}$ transforms like a spinor, that is, if $\psi \to \mathbf{S}\psi$, then $\psi^{(c)} \to \mathbf{S}\psi^{(c)}$. (2P)

Note: In the Majorana representation all four γ-matrices are purely imaginary, hence $\mathbf{C} = \mathbb{1}$ and $\psi^{(c)} = \psi$. That is, Majorana fermions are invariant under charge conjugation since they are not charged.

Exercise 12.2: Relativistic Hamilton equations of motion (6P)

Consider a particle in a 1+1-dimensional Minkowski space with coordinates $\mathbf{x} = (ct, x)$. Let us parameterize this space time by the coordinate time t, where the dot denotes the derivative with respect to t. Let $\xi(t) = \big(x(t), p(t)\big)^T$ be the trajectory of the particle in phase space according to the equations of motion with a given Hamilton function $H(x,p)$. Moreover, let us define $f \circ g = \{g, f\}$ as a compact notation for the Poisson bracket.

(a) Show that the equations of motion can be written as $\dot{\xi} = H \circ \xi$. (1P)

(b) Prove that for a given initial condition $\xi(0)$, the formal solution of the equations of motion is given by $\xi(t) = U(t)\,\xi\big|_{\xi=\xi(0)}$ with $U(t) = e^{tH\circ}$. (2P)

(c) Now let us consider the Hamilton function of the relativistic harmonic oscillator

$$H(x,p) = T(p) + V(x)$$

with $T(p) = c\sqrt{p^2 + m^2c^2}$ and $V(x) = \frac{1}{2}kx^2$. In non-relativistic mechanics, we can of course solve thi problem exactly. In the relativistic case, however, this is not so

[1]Here the star $\star$ denotes complex conjugation while the dagger † denotes complex conjugation plus transposition (called the adjoint or the Hermitean conjugate).

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easy because the equations of motion become nonlinear. Here one often uses the so-called *operator splitting approximation* $U(t) \approx e^{\frac{1}{2}tT\circ} e^{tV\circ} e^{\frac{1}{2}tT\circ}$. In addition, let us approximate the exponential functions to first order, i.e.

$$U(t) \;\to\; \tilde{U}(t) = \Big(1 + \frac{1}{2} tT\circ\Big)\Big(1 + tV\circ\Big)\Big(1 + \frac{1}{2} tT\circ\Big)\,.$$

Find the approximated solution $\tilde{\xi}(t) = \tilde{U}(t)\,\xi\big|_{\xi=\xi(0)}$. (2P)

(d) In numerical physics, this splitting scheme is used to approximate $U(t) \approx \big[\tilde{U}(t/N)\big]^N$ which becomes exact in the limit $N \to \infty$. Use *Mathematica*® or a similar software to plot $\tilde{\xi}_n = \big[\tilde{U}(1/N)\big]^n \xi(0)$ in phase space for $n = 0, 1, ..., 200$, using the following parameters and intial conditions (1P)

$$N = 10\,, \quad x(0) = 0\,, \quad p(0) = 10\,, \quad c = 1\,, \quad m = 1\,, \quad k = 1\,.$$

As you will see, the orbit in phase space is periodically closed but not circular.

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Sunday, July 09 at 23:59.