SAMPLE SOLUTIONS EXERCISE 9

EXERCISE 9.1: BOOST OF A NON-RELATIVISTIC WAVE FUNCTION (4P)

Consider a wave function $\psi(\mathbf{x}, t)$ which solves the non-relativistic Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{x}, t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\mathbf{x}, t).$$

The purpose of this exercise is to study its behavior under Galilei transformation from a rest frame S with $\{\mathbf{x}, t, \nabla, \partial_t\}$ to a frame $\tilde{S}$ moving at velocity $\mathbf{v}$ described by $\{\tilde{\mathbf{x}}, \tilde{t}, \tilde{\nabla}, \tilde{\partial}_t\}$.

(a) Show that $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) := \psi(\mathbf{x}, t)$ does *not* satisfy the Schrödinger equation in $\tilde{S}$. (1P)

(b) Use the fully general ansatz $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) := e^{i\phi(\mathbf{x},t)}\psi(\mathbf{x}, t)$ to derive a partial differential equation for the function $\phi(\mathbf{x}, t)$ in such a way that $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t})$ does satisfy the Schrödinger equation in $\tilde{S}$. (1P)

(c) Show that this ansatz holds for any $\psi(\mathbf{x}, t)$ only if $\phi(\mathbf{x})$ obeys the differential equations (1P)

$$\nabla\phi(\mathbf{x}, t) = -\frac{m\mathbf{v}}{\hbar}, \quad \partial_t\phi(\mathbf{x}, t) = \frac{mv^2}{2\hbar}.$$

Find the solution for $\phi(\mathbf{x}, t)$ and derive the boosted wave function $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t})$. (1P)

SAMPLE SOLUTION

The Galilei transformation reads (see lecture notes)

$$\tilde{x} = x - \mathbf{v}t\,, \quad \tilde{t} = t\,, \quad \tilde{\nabla} = \nabla\,, \quad \tilde{\partial}_t = \partial_t + \mathbf{v}\cdot\nabla$$

(a) We insert the naive ansatz $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) := \psi(\mathbf{x}, t)$ into the transformed Schrödinger equation: (1P)

$$\Big(i\hbar\tilde{\partial}_t + \frac{\hbar^2\tilde{\nabla}^2}{2m}\Big)\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) \;=\; \Big(i\hbar(\partial_t + \mathbf{v}\cdot\nabla) + \frac{\hbar^2\nabla^2}{2m}\Big)\psi(\mathbf{x}, t) \;=\; i\hbar\mathbf{v}\cdot\nabla\psi(\mathbf{x}, t) \neq 0.$$

(b) If we insert the fully general ansatz $\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) := e^{i\phi(\mathbf{x},t)}\psi(\mathbf{x}, t)$ into the transformed Schrödinger equation we get

$$\Big(i\hbar\tilde{\partial}_t + \frac{\hbar^2\tilde{\nabla}^2}{2m}\Big)\tilde{\psi}(\tilde{\mathbf{x}}, \tilde{t}) \;=\; \Big(i\hbar(\partial_t + \mathbf{v}\cdot\nabla) + \frac{\hbar^2\nabla^2}{2m}\Big)e^{i\phi(\mathbf{x},t)}\psi(\mathbf{x}, t) \;=$$

$$-\frac{he^{i\phi}}{2m}\Big(-2im\partial_t\psi - 2i\big(m\mathbf{v} + h\nabla\phi\big)\cdot\nabla\psi + \big(2m\partial_t\phi + 2m\mathbf{v}\cdot\nabla\phi + h(\nabla\phi)^2 - ih\nabla^2\phi\big)\psi - h\nabla^2\psi\Big)$$

Since $\psi(\mathbf{x}, t)$ obeys the Schrödinger equation in the rest frame, the first and the last term in the bracket cancels. Hence we are left with the partial differential equation (1P)

$$-2i\big(m\mathbf{v} + h\nabla\phi\big)\cdot\nabla\psi + \big(2m\partial_t\phi + 2m\mathbf{v}\cdot\nabla\phi + h(\nabla\phi)^2 - ih\nabla^2\phi\big)\psi = 0$$

(c) The equation derived above can hold for *any* $\psi(\mathbf{x},t)$ only if the two brackets vanish separately. The first bracket yields

$$\nabla\phi(\mathbf{x},t) = -\frac{m}{\hbar}\mathbf{v} = const.$$

With this result the second bracket simplifies to

$$\left(2m\partial_t\phi - 2m\mathbf{v}\cdot\frac{m}{\hbar}\mathbf{v} + \hbar\frac{m^2v^2}{\hbar^2} - ih\underbrace{\nabla^2\phi}_{=0}\right) = \left(2m\partial_t\phi - \frac{m^2v^2}{\hbar^2}\right) = 0.$$

Hence we end up with (1P)

$$\nabla\phi(\mathbf{x},t) = -\frac{m\mathbf{v}}{\hbar} = const\,, \quad \partial_t\phi(\mathbf{x},t) = \frac{mv^2}{2\hbar} = const$$

with the unique solution that $\phi(\mathbf{x},t)$ must be the following linear function:

$$\phi(\mathbf{x},t) = -\frac{m}{\hbar}\mathbf{v}\cdot\mathbf{x} + \frac{m}{2\hbar}v^2t\,.$$

Therefore, the boosted wave function reads (1P)

$$\tilde{\psi}(\tilde{\mathbf{x}},\tilde{t}) = \exp\left(-\frac{im\mathbf{v}\cdot\mathbf{x}}{\hbar}+\frac{imv^2t}{2\hbar}\right)\psi(\mathbf{x},t) = \exp\left(-\frac{im\mathbf{v}\cdot\tilde{\mathbf{x}}}{\hbar}-\frac{imv^2\tilde{t}}{2\hbar}\right)\psi(\tilde{\mathbf{x}}+\mathbf{v}\tilde{t},\tilde{t})$$

Exercise 9.2: Properties of the γ-matrices (4P)

The Dirac γ-matrices are defined by the anticommutation relations

$$\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1}\ ,$$

where $\mathbb{1}$ is the unit operator in spinor space. Prove the following properties using these relations, but **without using an explicit matrix representation** (the trace 'Tr[...]' is taken in spinor space. Recall that we are using the 'mostly plus' convention).

(a) $\gamma^\mu\gamma_\mu = -4\mathbb{1}$

(b) $\gamma^\mu\gamma^\nu\gamma_\mu = 2\gamma^\nu$

(c) $\mathrm{Tr}[\gamma^\mu] = 0.$ ← Hint: Show that $\gamma^5\gamma^5 = \mathbb{1}$ and consider $\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5]$

(d) $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4\eta^{\mu\nu}$

Sample Solution

(a) The contraction of two γ matrix vectors yields

$$\gamma^\mu\gamma_\mu = \frac{1}{2}\left(\eta_{\mu\nu}\gamma^\mu\gamma^\nu + \eta_{\mu\nu}\gamma^\nu\gamma^\mu\right) = \frac{1}{2}\eta_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} = \frac{1}{2}(-2)\underbrace{\eta_{\mu\nu}\eta^{\mu\nu}}_{=4} = -4\,\mathbb{1}$$

(b) If a single matrix γ^ν is enclosed by two contracted γ-matrices $\gamma^\mu \ldots \gamma_\mu$ we get:

$$\gamma^\mu\gamma^\nu\gamma_\mu \;=\; \gamma^\mu\big(\underbrace{\gamma^\nu\gamma_\mu + \gamma_\mu\gamma^\nu}_{=-2\delta^\nu_\mu} - \gamma_\mu\gamma^\nu\big) \;=\; -2\gamma^\nu - \underbrace{\gamma^\mu\gamma_\mu}_{=-4\mathbb{1}}\gamma^\nu \;=\; +2\gamma^\nu.$$

(c) The trace of a γ-matrix vanishes:

$$\mathrm{Tr}[\gamma^\mu] \;=\; \mathrm{Tr}[\gamma^\mu \underbrace{\gamma^5\gamma^5}_{=\mathbb{1}}] \;=\; -\mathrm{Tr}[\gamma^5\gamma^\mu\gamma^5] \;=\; \mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5] \;=\; -\mathrm{Tr}[\gamma^\mu]$$

$$\Rightarrow \quad \mathrm{Tr}[\gamma^\mu] \;=\; 0.$$

In the second equality we used $\{\gamma^\mu, \gamma^5\} = 0$. In the third equality we used that the trace is invariant under cyclic shifts of the matrices.

(d) The last relation can be proven as follows:

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] \;=\; \frac{1}{2}\Big(\mathrm{Tr}[\gamma^\mu\gamma^\nu] + \mathrm{Tr}[\gamma^\nu\gamma^\mu]\Big) \;=\; \frac{1}{2}\mathrm{Tr}\big[\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{=-2\eta^{\mu\nu}\mathbb{1}}\big] \;=\; -\mathrm{Tr}[\mathbb{1}]\eta^{\mu\nu} \;=\; -4\eta^{\mu\nu}.$$

⇒ **Please turn over**

EXERCISE 9.3: LORENTZ TRANSFORMATION OF DIRAC SPINORS (4P)

In this exercise we want to study plane-wave solutions of the Dirac equation

$$(i\gamma^\mu \partial_\mu - M)\psi(\mathbf{x}) = 0$$

where $M = mc/\hbar$ is the mass parameter and γ^μ denotes the standard Dirac representation of the γ-matrices.

(a) Insert the plane-wave ansatz $\psi(\mathbf{x}) = u_{\mathbf{p}} e^{\frac{i}{\hbar} p_\nu x^\nu}$, where $u_{\mathbf{p}}$ is a 4-component vector, the so-called *spinor*, which lives in the same space as the γ-matrices (spinor space), and where $\mathbf{p}$ is the 4-momentum of the plane wave. Derive a condition for $u_{\mathbf{p}}$. (1P)

(b) Consider a resting particle with $p^1 = p^2 = p^3 = 0$ and $p^0 = E/c$. Determine the possible spinors and the corresponding energies E. (1P)

(c) Perform a Lorentz boost in x-direction with the rapidity $\theta = \operatorname{arctanh}(v/c)$. Compute the transformed spinors $\tilde{u} = \mathbf{S}(\theta)u$ of the spinors determined in (b), where $\mathbf{S}(\theta)$ is the spinor transformation matrix derived in the Lecture Notes (you may use *Mathematica®* or similar software). (2P)

SAMPLE SOLUTION

(a) Inserting this ansatz yields

$$\Big(\frac{1}{\hbar}\gamma^\mu p_\mu + M\Big)u_{\mathbf{p}} = 0$$

(b) In this case the equation given above reduces to

$$\Big(\frac{1}{\hbar}\gamma^0 \frac{E}{c} - M\Big)u_{\mathbf{p}} = 0$$

or equivalently

$$\gamma^0 E u_{\mathbf{p}} = mc^2 u_{\mathbf{p}}$$

In matrices this equation reads

$$\begin{pmatrix} E & & & \\ & E & & \\ & & -E & \\ & & & -E \end{pmatrix} \mathbf{u} = mc^2 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \mathbf{u}_p$$

So we get two spinors

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

for solutions with positive energy $E = mc^2$ and two

$$u^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

for negative energy $E = -mc^2$.

(c) According to the Lecture Notes, the generator in spinor space corresponding to a (01)-boost in x-direction is given by

$$\sigma_{(01)} = \frac{i}{2}[\gamma_0, \gamma_1] = \begin{pmatrix} & & & -i \\ & & -i & \\ & -i & & \\ -i & & & \end{pmatrix}.$$

The corresponding transformation matrix reads

$$\mathbf{S}(\theta) = \exp\left(\frac{i}{2}\theta\sigma_{(01)}\right) = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) & 0 & 0 & \sinh\left(\frac{\theta}{2}\right) \\ 0 & \cosh\left(\frac{\theta}{2}\right) & \sinh\left(\frac{\theta}{2}\right) & 0 \\ 0 & \sinh\left(\frac{\theta}{2}\right) & \cosh\left(\frac{\theta}{2}\right) & 0 \\ \sinh\left(\frac{\theta}{2}\right) & 0 & 0 & \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}$$

hence

$$\tilde{u}_1 = \begin{pmatrix} \cosh\left(\frac{\theta}{2}\right) \\ 0 \\ 0 \\ \sinh\left(\frac{\theta}{2}\right) \end{pmatrix}, \quad \tilde{u}_2 = \begin{pmatrix} 0 \\ \cosh\left(\frac{\theta}{2}\right) \\ \sinh\left(\frac{\theta}{2}\right) \\ 0 \end{pmatrix}, \quad \tilde{u}_3 = \begin{pmatrix} 0 \\ \sinh\left(\frac{\theta}{2}\right) \\ \cosh\left(\frac{\theta}{2}\right) \\ 0 \end{pmatrix}, \quad \tilde{u}_4 = \begin{pmatrix} \sinh\left(\frac{\theta}{2}\right) \\ 0 \\ 0 \\ \cosh\left(\frac{\theta}{2}\right) \end{pmatrix}$$

As expected, they reduce to the old ones for $\theta = 0$.

(Σ = 12P)