Sample Solutions Exercise 8

Exercise 8.1: Complex charged scalar field (12P)

Let us consider a complex field ϕ, ϕ^* with the Lagrangian

$$\mathcal{L} = -(\partial_\mu \phi)^*(\partial^\mu \phi) = -(\partial_\mu \phi^*)(\partial^\mu \phi),$$

treating $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$ as independent fields. As one can see, this Lagrangian is invariant under global $U(1)$ transformations $\phi(\mathbf{x}) \to \tilde{\phi}(\mathbf{x}) = \phi(\mathbf{x})e^{i\theta}$, where θ does not depend on $\mathbf{x}$.

(a) Prove that the corresponding Noether current is $j^\mu = i\phi^*(\partial^\mu \phi) - i\phi(\partial^\mu \phi)^*$. (2P)

(b) Now let us instead consider *local* $U(1)$ transformations $\phi(\mathbf{x}) \to \tilde{\phi}(\mathbf{x}) = \phi(\mathbf{x})e^{i\theta(\mathbf{x})}$, where $\theta(\mathbf{x})$ is now a function of $\mathbf{x}$. Confirm that the Lagrangian $\mathcal{L} = -(\partial_\mu \phi)^*(\partial^\mu \phi)$ is generally *not* invariant under such local transformations. (1P)

(c) Replace the partial derivatives in the Lagrangian by so-called *covariant derivatives*

$$\partial^\mu \to D^\mu = \partial^\mu - iA^\mu(\mathbf{x}),$$

where $A^\mu(\mathbf{x})$ is the electromagnetic potential. Find a transformation

$$A^\mu(\mathbf{x}) \to \tilde{A}^\mu(\mathbf{x}) = A^\mu(\mathbf{x}) + f^\mu(\mathbf{x})$$

such that for given $\theta(\mathbf{x})$ the Lagrangian $\mathcal{L} = -(D_\mu \phi)^*(D^\mu \phi)$ is invariant under local $U(1)$ transformations. How can we interpret the transformation of $A^\mu(\mathbf{x})$? (3P)

(d) Consider now the combined Lagrangian

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - (D_\mu \phi)^*(D^\mu \phi)$$

and determine the equations of motion for ϕ, ϕ^*, and A_ν in Lorenz gauge. (3P)

(e) Show that the Noether current of this Lagrangian with respect to local $U(1)$ transformations is now given by $J^\mu = i\phi^*(D^\mu \phi) - i\phi(D^\mu \phi)^*$, reducing the equation of motion for the vector potential to $\Box A^\mu = \mu_0 J^\mu$. (2P)

(f) Verify that the Noether current is conserved, that is, $\partial_\mu J^\mu = 0$. To this end evaluate $\partial_\mu J^\mu$ and insert the equations of motion obtained in (d). (1P)

Sample Solution

(a) Global phase shifts $\phi \to \tilde{\phi} = \phi e^{i\theta}$ and $\phi^* \to \tilde{\phi}^* = \phi^* e^{-i\theta}$ are obviously a symmetry transformation of the Lagrangian $\mathcal{L} = -(\partial_\mu \phi^*)(\partial^\mu \phi)$, since these phase factors simply drop out. So we can directly apply the Noether theorem with $\Lambda = 0$. For infinitesimal $\theta = \epsilon \ll 1$, the field changes are (1P)

$$\Delta = i\phi\,, \qquad \Delta^* = -i\phi^*.$$

The corresponding conjugate momentum fields read

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -(\partial^\mu \phi)^*\,, \qquad \pi^{*\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = -\partial^\mu \phi.$$

Therefore the Noether current is given by (1P)

$$j^\mu \;=\; \pi^\mu \Delta + \pi^{*\mu}\Delta^* \;=\; -i\phi(\partial^\mu\phi)^* + i\phi^*(\partial^\mu\phi).$$

(b) The Lagrangian transforms as follows (suppressing the arguments $(\mathbf{x})$ for $\theta(\mathbf{x})$ and $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$: (1P)

$$\begin{aligned}
-(\partial_\mu\phi)^*(\partial^\mu\phi) &\rightarrow -(\partial_\mu \phi^* e^{-i\theta})(\partial^\mu \phi e^{i\theta}) \\
&= -\Big[(\partial_\mu\phi^*) - i\phi^*(\partial_\mu\theta)\Big]e^{-i\theta}\Big[(\partial^\mu\phi) + i\phi(\partial^\mu\theta)\Big]e^{i\theta} \\
&= -(\partial_\mu\phi)^*(\partial^\mu\phi) - i\phi(\partial^\mu\theta)(\partial_\mu\phi^*) + i\phi^*(\partial_\mu\theta)(\partial^\mu\phi) - \phi\phi^*(\partial_\mu\theta)(\partial^\mu\theta).
\end{aligned}$$

The last three terms are generally nonzero. For example, if $\phi = \phi^* = 1$, all terms would vanish except for the last one.

(c) Suppose that $\mathbf{A}(\mathbf{x}) \rightarrow \tilde{\mathbf{A}}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{f}(\mathbf{x})$, where the fields $\mathbf{A}$ and $\mathbf{f}$ are real-valued. Then the covariant derivative changes by

$$D_\mu = \partial_\mu - i\,A_\mu \;\rightarrow\; \tilde{D}_\mu = \partial_\mu - i\,A_\mu - i\,f_\mu.$$

We repeat the calculation with the covariant derivative: (1P)

$$\begin{aligned}
-(D_\mu\phi)^*(D^\mu\phi) &\rightarrow -(\tilde{D}_\mu\phi e^{i\theta})^*\,(\tilde{D}^\mu\phi\, e^{i\theta}) \\
&= -\big[(\partial_\mu + iA_\mu + if_\mu)\phi^* e^{-i\theta}\big]\,\big[(\partial^\mu - iA^\mu - if^\mu)\phi e^{i\theta}\big] \\
&= -\Big[(\partial_\mu\phi)^* - i\phi^*(\partial_\mu\theta) + i\phi^* A_\mu + i\phi^* f_\mu\Big]e^{-i\theta}\Big[(\partial^\mu\phi) + i\phi(\partial^\mu\theta) - i\phi A^\mu - i\phi f^\mu\Big]e^{i\theta} \\
&= -\Big[(D_\mu\phi)^* - i\phi^*(\partial_\mu\theta) + i\phi^* f_\mu\Big]\,\Big[(D^\mu\phi) + i\phi(\partial^\mu\theta) - i\phi f^\mu\Big]
\end{aligned}$$

Obviously, invariance can be established if $f_\mu = \partial_\mu\theta$. To summarize, the Lagrangian with the covariant derivative is invariant under the combined transformation (1P)

$$\phi(\mathbf{x}) \rightarrow \tilde{\phi}(\mathbf{x}) = \phi(\mathbf{x})e^{i\theta(\mathbf{x})}\,, \qquad A_\mu(\mathbf{x}) \rightarrow \tilde{A}_\mu(\mathbf{x}) = A_\mu(\mathbf{x}) + \partial_\mu\theta(\mathbf{x}).$$

The extra field $A^\mu(\mathbf{x})$ can be interpreted as an electromagnetic field which changes by the gradient of a scalar function $\partial_\mu\theta$, hence the transformation given above is just a gauge transformation. (1P)

(d) We first write out the Lagrangian:

$$\begin{aligned}
\mathcal{L} \;&=\; -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \;-\; \big((\partial_\mu\phi)^* + iA_\mu\phi^*\big)\big((\partial^\mu\phi) - iA^\mu\phi\big) \\
&=\; -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \;-\; (\partial_\mu\phi)^*(\partial^\mu\phi) \;+\; iA^\mu\phi(\partial_\mu\phi)^* \;-\; iA^\mu\phi^*(\partial_\mu\phi) \;-\; A_\mu A^\mu\phi\phi^*
\end{aligned}$$

Then we compute the equation of motion with respect to ϕ^*: (1P)

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} &= -iA^\mu(\partial_\mu\phi) - A_\mu A^\mu\phi + \partial_\mu\Big((\partial^\mu\phi) - iA^\mu\phi\Big) \\ &= -iA^\mu(\partial_\mu\phi) - A_\mu A^\mu\phi + \Box\phi - i(\partial_\mu A^\mu)\phi - iA^\mu(\partial_\mu\phi) \;=\; 0\end{aligned}$$

In Lorenz gauge we may set $\partial_\mu A^\mu = 0$. Thus the equation of motion reads

$$\Box\phi \;=\; 2iA^\mu(\partial_\mu\phi) + A_\mu A^\mu\phi + i\underbrace{(\partial_\mu A^\mu)}_{=0}\phi$$

The other equation with respect to ϕ is just the complex conjugate: (1P)

$$\Box\phi^* \;=\; -2iA^\mu(\partial_\mu\phi)^* + A_\mu A^\mu\phi^* - i\underbrace{(\partial_\mu A^\mu)}_{=0}\phi^*.$$

What remains is to find the equation of motion of A^ν: (1P)

$$\frac{\partial\mathcal{L}}{\partial A^\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)} = \frac{1}{\mu_0}\Big(\Box A_\nu - \partial_\nu\underbrace{(\partial_\mu A^\mu)}_{=0}\Big) + i\phi(\partial_\nu\phi)^* - i\phi^*(\partial_\nu\phi) - 2A_\nu\phi\phi^* = 0$$

$$\Rightarrow \quad \Box A_\nu \;=\; \mu_0\big(j_\nu + 2A_\nu\phi\phi^*\big) \quad \Rightarrow \quad \Box A^\nu \;=\; \mu_0\big(j^\nu + 2A^\nu\phi\phi^*\big)$$

(e) The Noether current can be computed in the same way as in (a), we only have to recalculate the conjugate momentum fields (1P)

$$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = -(\partial^\mu\phi)^* - iA^\mu\phi^* = (D^\mu\phi)^*$$

$$\Pi^{*\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} = -\partial^\mu\phi + iA^\mu\phi = (D^\mu\phi).$$

Note that we do not need to take the vector field $\mathbf{A}$ into account because its conjugate momentum vanishes. With $\Delta = i\phi$ and $\Delta^* = -i\phi^*$ we get (1P)

$$J^\mu \;=\; \Pi^\mu\Delta + \Pi^{*\mu}\Delta^* \;=\; -i\phi(D^\mu\phi)^* + i\phi^*(D^\mu\phi) \;=\; j^\mu + 2A^\mu\phi\phi^*$$

Therefore, the equation of motion for $\mathbf{A}$ simplifies to

$$\Box A^\mu = \mu_0 J^\mu.$$

(f) To verify that the Noether current is indeed conserved we compute

$$\begin{aligned}\partial_\mu J^\mu &= \partial_\mu\Big(i\phi^*(\partial^\mu\phi) - i\phi(\partial^\mu\phi)^* + 2A^\mu\phi\phi^*\Big) \\ &= i(\phi^*\Box\phi - \phi\Box\phi^*) + 2\underbrace{\partial_\mu A^\mu}_{=0}\phi\phi^* + 2A^\mu(\partial_\mu\phi)\phi^* + 2A^\mu\phi(\partial_\mu\phi)^*\end{aligned}$$

Inserting the equations of motion for ϕ and ϕ^* determined in (d) we get (1P)

$$i(\phi^*\Box\phi - \phi\Box\phi^*) = -2A^\mu(\partial_\mu\phi)^*\phi - iA_\mu A^\mu\phi\phi^* - 2A^\mu(\partial_\mu\phi)\phi^* + iA_\mu A^\mu\phi\phi^*,$$

where the second and the fourth term cancel. Inserting this expression above we get

$$\partial_\mu J^\mu = 0$$

meaning that the current is indeed conserved.

(Σ = 12P)