SAMPLE SOLUTIONS EXERCISE 7

EXERCISE 7.1: MINIMAL COUPLING AND COVARIANT DERIVATIVE (6P)

In this exercise we consider a non-relativistic classical and quantum particle in a given time-independent electromagnetic field $\phi(\mathbf{q}), \vec{A}(\mathbf{q})$, where $\mathbf{q} = \vec{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$. The aim is to illustrate the concept of *minimal coupling*. Starting point is the Lagrange function

$$L(\mathbf{q}, \dot{\mathbf{q}}) \;=\; \frac{1}{2}m\dot{\mathbf{q}}^2 - e\phi(\mathbf{q}) + e\dot{\mathbf{q}} \cdot \vec{A}(\mathbf{q})\,.$$

(a) Compute the generalized momentum $\mathbf{p}$. (1P)

(b) Verify that the Lagrange function is correct by determining the equations of motion and confirming that they are equivalent to the Lorentz force $\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$, where e is the electric charge, $\vec{v} = \dot{\mathbf{q}}$, $\vec{E} = -\mathrm{grad}(\phi) - \frac{\partial \vec{A}}{\partial t}$, and $\vec{B} = \mathrm{rot}(\vec{A})$. (2P)

(c) Compute the Hamilton function $H(\mathbf{q}, \mathbf{p})$ by the usual Legendre transformation and show that it is given by (2P)

$$H(\mathbf{q}, \mathbf{p}) \;=\; \frac{1}{2m}\Big(\mathbf{p} - e\vec{A}(\mathbf{q})\Big)^2 + e\phi(\mathbf{q})$$

> **Remark:** This boils down to a universal "cooking recipe" how to couple a system to an electromagnetic field which is known in the literature as *minimal coupling*: Simply add the potential $e\phi$ and replace the generalized momentum $\vec{p} \to \vec{p} - e\vec{A}$.

(d) Finally consider the quantum case and show that the Hamiltonian is given by

$$\mathbf{H} = -\frac{\hbar^2}{2m}\sum_{j=1}^{3} \mathbf{D}_j^2 + e\phi(\mathbf{q})$$

where $\mathbf{D}_j = \partial_j - \frac{ie}{\hbar}A_j$ is the so-called *covariant derivative*. (1P)

SAMPLE SOLUTION

(a) The generalized momentum $\mathbf{p} = (p_1, p_2, p_3)$ is given by the components

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i + eA_i(\mathbf{q}) \quad \Rightarrow \quad \mathbf{p} = m\dot{\mathbf{q}} + e\vec{A}(\mathbf{q})$$

(b) First we write down the Lorentz force:

$$\vec{F} = e\Big(\vec{E} + \vec{v} \times \vec{B}\Big) = e\Big(-\vec{\nabla}\phi - \underbrace{\frac{\partial \vec{A}(\mathbf{q})}{\partial t}}_{=0} + \vec{v} \times (\vec{\nabla} \times \vec{A}(\mathbf{q}))\Big)$$

Applying the Graßmann identity (bac-cab rule) we get

$$\Rightarrow \quad \vec{F} = e\Big(-\vec{\nabla}\phi + \vec{\nabla}(\vec{v} \cdot \vec{A}(\mathbf{q})) - (\vec{v} \cdot \vec{\nabla})\vec{A}(\mathbf{q})\Big)\,.$$

Finally we write it down in terms of components (1P)

$$F_i = m\dot{q}_i = e\Big(-\frac{\partial \phi}{\partial q_i} + \frac{\partial}{\partial q_i}\sum_{j=1}^{3} \dot{q}_j A_j(\mathbf{q}) - \sum_{j=1}^{3} \dot{q}_j \frac{\partial}{\partial q_j} A_i(\mathbf{q})\Big) \tag{1}$$

Now we turn to the Lagrange equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial \dot{q}_i} = \frac{\partial L(\mathbf{q},\dot{\mathbf{q}})}{\partial q_i}\,,$$

where

$$L(\mathbf{q},\dot{\mathbf{q}}) \;=\; \frac{1}{2}m\dot{\mathbf{q}}^2 - e\phi(\mathbf{q}) + e\dot{\mathbf{q}}\cdot\vec{A}(\mathbf{q})$$

This gives

$$m\ddot{q}_i + \frac{\mathrm{d}}{\mathrm{d}t}\big(eA_i(\mathbf{q})\big) \;=\; -e\frac{\partial \phi}{\partial q_i} + e\sum_{j=1}^{3}\dot{q}_j\frac{\partial A_j(\mathbf{q})}{\partial q_i}$$

Expanding the total derivative via

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(eA_i(\mathbf{q})\big) \;=\; e\sum_{j=1}^{3}\frac{\partial A_i}{\partial q_j}\dot{q}_j$$

we arrive at the expression in (**??**). (1P)

(c) The Hamilton function is defined via Legendre transformation as

$$H(\mathbf{q},\mathbf{p}) \;=\; \mathbf{p}\cdot\dot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}})$$

Inserting the Lagrange function yields (1P)

$$H(\mathbf{q},\mathbf{p}) \;=\; \mathbf{p}\cdot\dot{\mathbf{q}} - \frac{1}{2}m\dot{\mathbf{q}}^2 + e\phi(\mathbf{q}) - e\dot{\mathbf{q}}\cdot\vec{A}(\mathbf{q})$$

To complete a Legendre transformation, one has to replace the "wrong" variable $\dot{\mathbf{q}}$ by inserting the generalized momentum obtained in (a) by $\dot{\mathbf{q}} = \frac{\mathbf{p}}{m} - \frac{e}{m}\vec{A}(\mathbf{q})$:

$$H(\mathbf{p},\mathbf{q}) = \frac{\mathbf{p}^2}{m} - \frac{e}{m}\mathbf{p}\cdot\vec{A} - \frac{m}{2}\Big(\frac{\mathbf{p}^2}{2m} - \frac{2e\mathbf{p}\cdot\vec{A}}{m^2} + \frac{e^2}{m^2}\vec{A}^2\Big) + e\phi - \frac{e}{m}\mathbf{p}\cdot\vec{A} + \frac{e^2}{m}\vec{A}^2.$$

Collecting all terms this simplifies to (1P)

$$H(\mathbf{p},\mathbf{q}) = \underbrace{\frac{\mathbf{p}^2}{2m} - \frac{e}{m}\mathbf{p}\cdot\vec{A} + \frac{e^2}{2m}\vec{A}^2}_{\frac{1}{2m}(\mathbf{p}-e\vec{A})^2} + e\phi.$$

(d) The quantum Hamiltonian $\mathbf{H}$ in the position representation can be obtained via the Bohr rule by replacing $\mathbf{p} \to -i\hbar\vec{\nabla}$: (1P)

$$\mathbf{H} = \frac{1}{2m}\Big(-i\hbar\vec{\nabla}^2 - e\vec{A}(\mathbf{q})\Big)^2 + e\phi(\mathbf{q})$$

$$= -\frac{\hbar^2}{2m}\underbrace{\Big(\vec{\nabla} - \frac{ie}{\hbar}\vec{A}(\mathbf{q})\Big)^2}_{\sum_{j=1}^{3}\mathbf{D}_j^2} + e\phi(\mathbf{q})$$

EXERCISE 7.2: KALUZA-KLEIN THEORY (6P)

Standard (quantum) electrodynamics regards the $U(1)$ circle as a separate additional structure attached to space-time. A very interesting alternative approach is the so-called Kaluza-Klein theory, which interprets the $U(1)$ circle as an additional fifth dimension, integrating it into space-time and applying general relativity (see lecture notes). With this exercise we want to get into touch with some aspects of the *Kaluza-Klein miracle.*

(a) Consider a garden hose with coordinate x and a compactified coordinate $\phi \in [0, 2\pi]$. Let us for simplicity ignore the periodicity, treating ϕ as if $\phi \in \mathbb{R}$ was unrestricted. Suppose that without a twist, the metric tensor on the hose is given by

$$g_{ij} = \begin{pmatrix} g_{xx} & g_{x\phi} \\ g_{\phi x} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & \\ & \rho^2 \end{pmatrix},$$

where ρ is the radius of the tube. Now let us twist the garden hose uniformly, as shown in the figure above. This will correspond to a coordinate transformation $\tilde{x} = x$, $\tilde{\phi} = \phi - ax$, where $-a$ is the slope of the orange line in the figure. Compute the metric tensor $\tilde{g}_{ij}$ in the coordinates $\tilde{x}, \tilde{\phi}$. (2P)

(b) Repeat part (a), but now applied to a 1+3+1-dimensional garden hose with coordinates $(x^0, x^1, x^2, x^3, x^4) = (ct, x, y, z, \phi)$ and the metric $g_{ij} = \mathrm{diag}(-1, 1, 1, 1, \rho^2)$. Let us use Latin indices for the range 0...4 and Greek indices for the range 0...3. Suppose that the garden hose is deformed uniformly by $\tilde{x}^\mu = x^\mu$, $\tilde{x}^4 = x^4 - \frac{e}{\hbar} A_\mu x^\mu$, where e is the elementary charge and A^μ is the electromagnetic 4-potential which is assumed here to be constant. Compute $\tilde{g}_{ij}$ in the coordinates $\tilde{x}^i$. (1P)

(c) Calculate the determinant $\tilde{g} = \det(\tilde{g}_{ij})$ and show that the inverse $\tilde{g}^{ij}$ in block notation is given by (1P)

$$\tilde{g}^{ij} = \left(\begin{array}{c|c} \eta^{\mu\nu} & -\frac{e}{\hbar} A^\mu \\ \hline -\frac{e}{\hbar} A^\nu & \frac{e^2}{\hbar^2} A_\sigma A^\sigma + \rho^{-2} \end{array} \right).$$

(d) A central idea of the Kaluza-Klein theory is that motion along the extra dimension is nothing but electric charge. Quantum-mechanically, as the extra dimension $\phi = x^4$ is compactified modulo 2π, this motion is quantized, explaining nicely the observed quantization of electric charge. Let us play a little bit with these thoughts by considering a factorized wave function

$$\Psi(x^0, \ldots, x^4) = \psi(x^0, \ldots, x^3) e^{in\phi},$$

where $n \in \mathbb{Z}$ counts the number of elementary charges. In the following we would like to solve the massless wave equation $\Box\Psi = 0$ in the geometry given by $\tilde{g}$. However, since $\tilde{g}^{ij}$ is non-diagonal, we have to replace $\Box = \partial_i \partial^i$ by the so-called *Laplace-Beltrami operator* (cf. previous exercise)

$$\Box_{\tilde{g}} = \frac{1}{\sqrt{|\tilde{g}|}} \partial_i \sqrt{|\tilde{g}|} \tilde{g}^{ij} \partial_j .$$

Write out the wave equation $\Box_{\tilde{g}} \Psi = 0$ in components. As you will see, the Kaluza-Klein theory is consistent with the principle of minimal coupling. (2P)

Sample Solution

(a) First we invert the coordinate transformation $\tilde{x} = x$, $\tilde{\phi} = \phi - ax$, getting

$$x = \tilde{x}\,, \quad \phi = \tilde{\phi} + a\tilde{x}.$$

Next we recall the transformation law for the metric: (1P)

$$\tilde{g}_{ij} \;=\; g_{k\ell}\frac{\partial x^k}{\partial \tilde{x}^j}\frac{\partial x^\ell}{\partial \tilde{x}^i}.$$

Remark: If you don't recall this transformation law, you can derive it easily from the invariance of the line element:

$$\mathrm{d}s^2 = \tilde{g}_{ij}\,\mathrm{d}\tilde{x}^i\,\mathrm{d}\tilde{x}^j = g_{k\ell}\frac{\partial x^k}{\partial \tilde{x}^j}\frac{\partial x^\ell}{\partial \tilde{x}^i}\,\mathrm{d}\tilde{x}^i\,\mathrm{d}\tilde{x}^j = g_{k\ell}\,\mathrm{d}x^k\,\mathrm{d}x^\ell\,.$$

Now we simply compute the matrix entries: (1P)

$$\tilde{g}_{\tilde{x}\tilde{x}} = \underbrace{g_{xx}}_{=1}\underbrace{\frac{\partial x}{\partial \tilde{x}}}_{=1}\underbrace{\frac{\partial x}{\partial \tilde{x}}}_{=1} + \underbrace{g_{\phi\phi}}_{=\rho^2}\underbrace{\frac{\partial \phi}{\partial \tilde{x}}}_{=a}\underbrace{\frac{\partial \phi}{\partial \tilde{x}}}_{=a} = 1 + \rho^2 a^2$$

$$\tilde{g}_{\tilde{x}\tilde{\phi}} = \tilde{g}_{\tilde{\phi}\tilde{x}} = \underbrace{g_{xx}}_{=1}\underbrace{\frac{\partial x}{\partial \tilde{x}}}_{=1}\underbrace{\frac{\partial x}{\partial \tilde{\phi}}}_{=0} + \underbrace{g_{\phi\phi}}_{=\rho^2}\underbrace{\frac{\partial \phi}{\partial \tilde{x}}}_{=a}\underbrace{\frac{\partial \phi}{\partial \tilde{\phi}}}_{=1} = \rho^2 a$$

$$\tilde{g}_{\tilde{\phi}\tilde{\phi}} = \underbrace{g_{xx}}_{=1}\underbrace{\frac{\partial x}{\partial \tilde{\phi}}}_{=0}\underbrace{\frac{\partial x}{\partial \tilde{\phi}}}_{=0} + \underbrace{g_{\phi\phi}}_{=\rho^2}\underbrace{\frac{\partial \phi}{\partial \tilde{\phi}}}_{=1}\underbrace{\frac{\partial \phi}{\partial \tilde{\phi}}}_{=1} = \rho^2\,.$$

So altogether the metric reads

$$\tilde{g}_{ij} = \begin{pmatrix} \tilde{g}_{\tilde{x}\tilde{x}} & \tilde{g}_{\tilde{x}\tilde{\phi}} \\ \tilde{g}_{\tilde{\phi}\tilde{x}} & \tilde{g}_{\tilde{\phi}\tilde{\phi}} \end{pmatrix} = \begin{pmatrix} 1+\rho a^2 & \rho^2 a \\ \rho^2 a & \rho^2 \end{pmatrix}\,.$$

This matrix can be inverted easily, giving

$$\tilde{g}^{ij} = \begin{pmatrix} \tilde{g}^{\tilde{x}\tilde{x}} & \tilde{g}^{\tilde{x}\tilde{\phi}} \\ \tilde{g}^{\tilde{\phi}\tilde{x}} & \tilde{g}^{\tilde{\phi}\tilde{\phi}} \end{pmatrix} = \begin{pmatrix} 1 & -a \\ -a & a^2+\rho^{-2} \end{pmatrix}\,.$$

(b) The calculation is fully analogous to (a), only with more components. First we invert the transformation

$$x^\mu = \tilde{x}^\mu\,, \quad x^4 = \tilde{x}^4 + \frac{e}{\hbar}A_\mu \tilde{x}^\mu$$

Then we use the same transformation law to compute the matrix elements

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \rho^2\tfrac{e^2}{\hbar^2}A_\mu A_\nu$$

$$\tilde{g}_{\mu 4} = g_{4\mu} = \rho^2\tfrac{e}{\hbar}A_\mu$$

$$\tilde{g}_{44} = \rho^2$$

So the full metric (the so-called Kaluza-Klein metric) reads (1P)

$$\tilde{g}_{ij} = \begin{bmatrix} g_{\mu\nu} + \rho^2\frac{e^2}{\hbar^2}A_\mu A_\nu & \rho^2\frac{e}{\hbar}A_\mu \\ \rho^2\frac{e}{\hbar}A_\mu & \rho^2 \end{bmatrix}$$

or in full form

$$\tilde{g}_{ij} = \begin{pmatrix} -1+\rho^2\frac{e^2}{\hbar^2}A_0A_0 & \rho^2\frac{e^2}{\hbar^2}A_0A_1 & \rho^2\frac{e^2}{\hbar^2}A_0A_2 & \rho^2\frac{e^2}{\hbar^2}A_0A_3 & \rho^2\frac{e}{\hbar}A_0 \\ \rho^2\frac{e^2}{\hbar^2}A_1A_0 & 1+\rho^2\frac{e^2}{\hbar^2}A_1A_1 & \rho^2\frac{e^2}{\hbar^2}A_1A_2 & \rho^2\frac{e^2}{\hbar^2}A_1A_3 & \rho^2\frac{e}{\hbar}A_1 \\ \rho^2\frac{e^2}{\hbar^2}A_2A_0 & \rho^2\frac{e^2}{\hbar^2}A_2A_1 & 1+\rho^2\frac{e^2}{\hbar^2}A_2A_2 & \rho^2\frac{e^2}{\hbar^2}A_2A_3 & \rho^2\frac{e}{\hbar}A_2 \\ \rho^2\frac{e^2}{\hbar^2}A_3A_0 & \rho^2\frac{e^2}{\hbar^2}A_3A_1 & \rho^2\frac{e^2}{\hbar^2}A_3A_2 & 1+\rho^2\frac{e^2}{\hbar^2}A_3A_3 & \rho^2\frac{e}{\hbar}A_3 \\ \rho^2\frac{e}{\hbar}A_0 & \rho^2\frac{e}{\hbar}A_1 & \rho^2\frac{e}{\hbar}A_2 & \rho^2\frac{e}{\hbar}A_3 & \rho^2 \end{pmatrix}$$

(c) With *Mathematica®* or similar software we compute the determinant

$$\tilde{g} = \det(\tilde{g}_{ij}) = -\rho^2$$

as well as the inverse

$$\tilde{g}^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 & +\frac{e}{\hbar}A_0 \\ 0 & 1 & 0 & 0 & -\frac{e}{\hbar}A_1 \\ 0 & 0 & 1 & 0 & -\frac{e}{\hbar}A_2 \\ 0 & 0 & 0 & 1 & -\frac{e}{\hbar}A_3 \\ +\frac{e}{\hbar}A_0 & -\frac{e}{\hbar}A_1 & -\frac{e}{\hbar}A_2 & -\frac{e}{\hbar}A_3 & Q+\rho^{-2} \end{pmatrix},$$

where $Q = \frac{e^2}{\hbar^2}[-(A_0)^2 + (A_1)^2 + (A_2)^2 + (A_3)^2]$. Using $A^\mu = \eta^{\mu\nu}A_\nu$ we can consistently raise the index of the electromagnetic potential components and write $Q = \frac{e^2}{\hbar^2}A_\sigma A^\sigma$: (1P)

$$\tilde{g}^{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 & -\frac{e}{\hbar}A^0 \\ 0 & 1 & 0 & 0 & -\frac{e}{\hbar}A^1 \\ 0 & 0 & 1 & 0 & -\frac{e}{\hbar}A^2 \\ 0 & 0 & 0 & 1 & -\frac{e}{\hbar}A^3 \\ -\frac{e}{\hbar}A^0 & -\frac{e}{\hbar}A^1 & -\frac{e}{\hbar}A^2 & -\frac{e}{\hbar}A^3 & \frac{e^2}{\hbar^2}A_\sigma A^\sigma+\rho^{-2} \end{pmatrix} = \left(\begin{array}{c|c} \eta^{\mu\nu} & -\frac{e}{\hbar}A^\mu \\ \hline -\frac{e}{\hbar}A^\nu & \frac{e^2}{\hbar^2}A_\sigma A^\sigma+\rho^{-2} \end{array}\right)$$

(d) Since the metric is non-diagonal but constant, the determinant is also constant and hence the square roots in the Laplace-Beltrami operator cancel. So we are left with

$$\Box_{\tilde{g}} = \tilde{g}^{ij}\partial_i\partial_j = \partial_\mu\partial^\mu - \frac{2e}{\hbar}A^\mu\partial_\mu\partial_4 + \left(\frac{e^2}{\hbar^2}A_\sigma A^\sigma + \rho^{-2}\right)\partial_4\partial_4 .$$

Now we apply this operator to $\Psi = \psi e^{in\phi} = \psi e^{inx^4}$, where each application of the derivative ∂_4 simply brings down a factor in:

$$\Box_{\tilde{g}}\Psi = \left[\partial_\mu\partial^\mu - i\frac{2ne}{\hbar}A^\mu\partial_\mu - \left(\frac{e^2}{\hbar^2}A_\sigma A^\sigma + \rho^{-2}\right)n^2\right]\Psi.$$

We can rewrite it as

$$\Box_{\tilde{g}}\Psi = \left[\left(\partial_\mu - i\frac{ne}{\hbar}A_\mu\right)\left(\partial^\mu - i\frac{ne}{\hbar}A^\mu\right) - \rho^{-2}n^2\right]\Psi$$

Remark: This is part of the “Kaluza-Klein miracle”: The wave equation turns into a Klein-Gordon equation with the mass $m = n/\rho$. The motion in the circle, which is attributed to carrying an electric charge, gives the particle, although massless in 1+3+1 dimensions, an effective mass in 1+3 dimensions. In fact, in Nature we do not observe massless charged particles. Moreover, the Kaluza-Klein theory nicely reproduces the concept of minimal coupling studied in the previous exercise.

(Σ = 12P)