Sample Solutions Exercise 6

Exercise 6.1: Complex scalar field (4P)

Let us consider the following action for a pair of real-valued scalar fields $\psi(\mathbf{x})$ and $\chi(\mathbf{x})$:

$$S[\psi,\chi] \;=\; -\frac{1}{2}\int\Big(\eta^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu}+\eta^{\mu\nu}\frac{\partial\chi}{\partial x^\mu}\frac{\partial\chi}{\partial x^\nu}\Big)\,\mathrm{d}^4x$$

(a) Determine the equations of motion. (1P)

(b) Rewrite the action as $S[\phi,\phi^*]$ in terms of a complex-valued field $\phi=\frac{1}{\sqrt{2}}(\chi+i\psi)$ and $\phi^*=\frac{1}{\sqrt{2}}(\chi-i\psi)$. (1P)

(c) Find the equations of motion for $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$ (treating ϕ and ϕ^* as independent) and confirm that they are compatible with (a). (1P)

(d) Extend the action $S[\phi,\phi^*]$ by a mass term and derive the equations of motion. (1P)

Sample Solution

(a) For the Lagrange density $\mathcal{L}(\psi,\chi)=-\frac{1}{2}(\partial^\mu\psi\partial_\mu\psi+\partial^\mu\chi\partial_\mu\chi)$ the equation of motion for ψ reads (1P)

$$\frac{\partial\mathcal{L}}{\partial\psi}-\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Big)=0 \quad\Rightarrow\quad \Box\psi=\partial_\mu\partial^\mu\psi=0.$$

The same equation of motion is obtained for χ, i.e., the two fields decouple entirely.

(b) First we invert the definition of the complex field:

$$\phi=\frac{1}{\sqrt{2}}(\chi+i\psi) \quad\Rightarrow\quad \chi=\frac{1}{\sqrt{2}}(\phi+\phi^*)\,,\quad \psi=\frac{1}{\sqrt{2}i}(\phi-\phi^*).$$

This allows us to rewrite the terms in the Lagrangian as

$$\partial^\mu\chi\partial_\mu\chi \;=\; \frac{1}{2}\Big(\partial^\mu\phi\partial_\mu\phi+2\partial^\mu\phi^*\partial_\mu\phi+\partial^\mu\phi^*\partial_\mu\phi^*\Big)$$

$$\partial^\mu\psi\partial_\mu\psi \;=\; \frac{1}{2}\Big(-\partial^\mu\phi\partial_\mu\phi+2\partial^\mu\phi^*\partial_\mu\phi-\partial^\mu\phi^*\partial_\mu\phi^*\Big)$$

and hence the Lagrangian and the corresponding action are given by (1P)

$$\mathcal{L}(\phi,\phi^*) \;=\; -\frac{1}{2}(\partial^\mu\psi\partial_\mu\psi+\partial^\mu\chi\partial_\mu\chi) \;=\; -\partial^\mu\phi\partial_\mu\phi^*$$

$$\Rightarrow\quad S[\phi,\phi^*] \;=\; -\int \mathrm{d}^4x\big(\partial^\mu\phi^*\partial_\mu\phi\big)$$

(c) As in complex analysis, we treat ϕ and ϕ^* as independent variables:

$$-\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Big) \;=\; \partial_\mu\partial^\mu\phi \;=\; \Box\phi = 0$$

and similarly for ϕ^*. So the equations of motion read (1P)

$$\Box\phi = 0\,, \qquad \Box\phi^* = 0.$$

Since ϕ and ϕ^* depend linearly on χ and ψ, this implies that $\Box\chi = \Box\psi = 0$ and vice versa.

(d) A natural extension of $S[\phi,\phi^*]$ by a mass term would be (1P)

$$S[\phi,\phi^*] = -\int \mathrm{d}^4x\big(\partial^\mu\phi^*\partial_\mu\phi + m^2\phi^*\phi\big)$$

Then the Lagrange equations of motion

$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Big) = 0$$

and similarly for ϕ turn into two copies of the Klein-Gordon equation

$$(\Box - m^2)\phi = 0\,, \qquad (\Box - m^2)\phi^* = 0.$$

Remark: As will be shown later, the complexification of a field amounts to giving it an electric charge. Here this is not yet visible, one has simply two decoupled field components, either χ,ψ language or in the form of ϕ and ϕ^*. The story becomes nontrivial only when the field is coupled to an electromagnetic field A_μ.

Exercise 6.2: Invariants of the electromagnetic field tensor (8P)

(a) A Lorentz transformation is an isometry in Minkowski space. Show that this implies that the determinant of a Lorentz transformation matrix $\Lambda^\mu{}_\nu$ equals ±1. (2P)

(b) Demonstrate that the 4-dimensional fully antisymmetric Levi-Civita symbols $\epsilon_{\alpha\beta\gamma\delta}$ with $\epsilon^{\alpha\beta\gamma\delta} := -\epsilon_{\alpha\beta\gamma\delta}$ are invariant under special Lorentz transformations $SO(1,3)$ while they may change their sign under general Lorentz transformations $O(1,3)$. Note: This is the reason why they are called *symbols* and not *tensor*. (3P)

(c) The dual field tensor of the electromagnetic field is defined as ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\tau}F^{\rho\tau}$. Compute ${}^\star F_{\mu\nu}$ as well as the three Lorentz-invariant quantities

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\,, \qquad \frac{1}{4}\,{}^\star F_{\mu\nu}F^{\mu\nu}\,, \qquad \frac{1}{2}\,{}^\star F_{\mu\nu}\,{}^\star F^{\mu\nu}$$

in terms of the fields $\vec{E}$ and $\vec{B}$. (2P)

(d) Is it possible to convert a pure electric field into a pure magnetic field by Lorentz transformation? (1P)

Sample Solution

(a) A Lorentz transformation Λ is an isometry in Minkowski space, meaning that the metric tensor remains invariant under Lorentz transformations: (1P)

$$\eta^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\tau}\eta^{\rho\tau}$$

This can also be written as

$$\eta^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\eta^{\rho\tau}(\Lambda^T)_{\tau}{}^{\nu}$$

which can be interpreted as a matrix equation $\eta = \Lambda\eta\Lambda^T$ (analogous to $OO^T = \mathbb{1}$ for ordinary orthogonal transformations). Hence (1P)

$$\det(\eta) = \det(\Lambda)\det(\eta)\det(\Lambda) \quad \Rightarrow \quad \det(\Lambda)\det(\Lambda^T) = 1$$

and since $\det(\Lambda) = \det(\Lambda^T)$ we arrive at

$$\det(\Lambda) = \pm 1.$$

(b) The Levi-Civita symbols transform as

$$\epsilon_{\alpha\beta\gamma\delta} \;\rightarrow\; \tilde{\epsilon}_{\alpha\beta\gamma\delta} = \Lambda_{\alpha}{}^{\mu}\Lambda_{\beta}{}^{\nu}\Lambda_{\gamma}{}^{\rho}\Lambda_{\delta}{}^{\tau}\epsilon_{\mu\nu\rho\tau}$$

Now it is easy to see that:

- If two of the indices $\alpha, \beta, \gamma, \delta$ coincide, say $\alpha = \beta$, then the antisymmetry of $\epsilon_{\mu\nu\rho\tau}$ in the indices μ, ν forces $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$ to be zero.
- Otherwise, a transposition of two of the indices $\alpha, \beta, \gamma, \delta$, say α and β, results effectively in a transposition of μ, ν, meaning that $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$.

Hence we already know that $\tilde{\epsilon}$ is totally antisymmetric as well, the only thing we have to show is $1 = \epsilon_{0123} = \tilde{\epsilon}_{0123}$. Here we realize that

$$\tilde{\epsilon}_{0123} = \underbrace{\Lambda_0{}^{\mu}\Lambda_1{}^{\nu}\Lambda_2{}^{\rho}\Lambda_3{}^{\tau}\epsilon_{\mu\nu\rho\tau}}_{=\det(\Lambda)}$$

Thus, the Levi Civita symbols are invariant under all Lorentz transformations with $\det(\Lambda) = 1$ (called special Lorentz transformations in SO(1,3)) while they change sign under transformations with $\det(\Lambda) = -1$, for example, under reflections.

(c) The dual field tensor turns out to be given by (1P)

$$^{\star}F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_x/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{pmatrix}$$

and accordingly

$$^{\star}F^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_x/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}.$$

This gives (1P)

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \vec{B}^2 - \frac{\vec{E}^2}{c^2}$$
$$\frac{1}{4}\,{}^{\star}F_{\mu\nu}F^{\mu\nu} = \frac{\vec{B}\cdot\vec{E}}{c}$$
$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \frac{\vec{E}^2}{c^2} - \vec{B}^2$$

(d) No, because $\vec{B}^2 - \frac{\vec{E}^2}{c^2}$ is invariant and thus cannot change sign.

(Σ = 12P)