Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2021

Quantum boost

Exercise 9.1: Boost of a non-relativistic wave function (4P)

Consider a wave function $\psi(\mathbf{x},t)$ which solves the non-relativistic Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{x},t) = -\frac{\hbar^2\nabla^2}{2m}\psi(\mathbf{x},t).$$

The purpose of this exercise is to study its behavior under Galilei transformation from a rest frame S with $\{\mathbf{x},t,\nabla,\partial_t\}$ to a frame $\tilde{S}$ moving at velocity $\mathbf{v}$ described by $\{\tilde{\mathbf{x}},\tilde{t},\tilde{\nabla},\tilde{\partial}_t\}$.

(a) Show that $\tilde{\psi}(\tilde{\mathbf{x}},\tilde{t}) := \psi(\mathbf{x},t)$ does *not* satisfy the Schrödinger equation in $\tilde{S}$. (1P)

(b) Use the fully general ansatz $\tilde{\psi}(\tilde{\mathbf{x}},\tilde{t}) := e^{i\phi(\mathbf{x},t)}\psi(\mathbf{x},t)$ to derive a partial differential equation for the function $\phi(\mathbf{x},t)$ in such a way that $\tilde{\psi}(\tilde{\mathbf{x}},\tilde{t})$ does satisfy the Schrödinger equation in $\tilde{S}$. (1P)

(c) Show that this ansatz holds for any $\psi(\mathbf{x},t)$ only if $\phi(\mathbf{x})$ obeys the differential equations (1P)

$$\nabla\phi(\mathbf{x},t) = -\frac{m\mathbf{v}}{\hbar}\,, \quad \partial_t\phi(\mathbf{x},t) = \frac{mv^2}{2\hbar}.$$

Find the solution for $\phi(\mathbf{x},t)$ and derive the boosted wave function $\tilde{\psi}(\tilde{\mathbf{x}},\tilde{t})$. (1P)

Exercise 9.2: Properties of the γ-matrices (4P)

The Dirac γ-matrices are defined by the anticommutation relations

$$\{\gamma^\mu,\gamma^\nu\} \;=\; -2\eta^{\mu\nu}\mathbb{1}\ ,$$

where $\mathbb{1}$ is the unit operator in spinor space. Prove the following properties using these relations, but **without using an explicit matrix representation** (the trace 'Tr[...]' is taken in spinor space. Recall that we are using the 'mostly plus' convention).

(a) $\gamma^\mu\gamma_\mu = -4\mathbb{1}$

(b) $\gamma^\mu\gamma^\nu\gamma_\mu = 2\gamma^\nu$

(c) $\mathrm{Tr}[\gamma^\mu] = 0$. ← Hint: Show that $\gamma^5\gamma^5 = \mathbb{1}$ and consider $\mathrm{Tr}[\gamma^\mu\gamma^5\gamma^5]$

(d) $\mathrm{Tr}[\gamma^\mu\gamma^\nu] = -4\eta^{\mu\nu}$

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Exercise 9.3: Lorentz transformation of Dirac spinors (4P)

In this exercise we want to study plane-wave solutions of the Dirac equation

$$(i\gamma^\mu \partial_\mu - M)\psi(\mathbf{x}) = 0$$

where $M = mc/\hbar$ is the mass parameter and γ^μ denotes the standard Dirac representation of the γ-matrices.

(a) Insert the plane-wave ansatz $\psi(\mathbf{x}) = u_{\mathbf{p}} e^{\frac{i}{\hbar} p_\nu x^\nu}$, where $u_{\mathbf{p}}$ is a 4-component vector, the so-called *spinor*, which lives in the same space as the γ-matrices (spinor space), and where $\mathbf{p}$ is the 4-momentum of the plane wave. Derive a condition for $u_{\mathbf{p}}$. (1P)

(b) Consider a resting particle with $p^1 = p^2 = p^3 = 0$ and $p^0 = E/c$. Determine the possible spinors and the corresponding energies E. (1P)

(c) Perform a Lorentz boost in x-direction with the rapidity $\theta = \operatorname{arctanh}(v/c)$. Compute the transformed spinors $\tilde{u} = \mathbf{S}(\theta)u$ of the spinors determined in (b), where $\mathbf{S}(\theta)$ is the spinor transformation matrix derived in the Lecture Notes (you may use *Mathematica*® or similar software). (2P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, June 18 at 12:00.