Electrically charged cat as a source field [Wikimedia].

Exercise 8.1: Complex charged scalar field (12P)

Let us consider a complex field ϕ, ϕ^* with the Lagrangian

$$\mathcal{L} = -(\partial_\mu \phi)^* (\partial^\mu \phi) = -(\partial_\mu \phi^*)(\partial^\mu \phi),$$

treating $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$ as independent fields. As one can see, this Lagrangian is invariant under global $U(1)$ transformations $\phi(\mathbf{x}) \to \tilde{\phi}(\mathbf{x}) = \phi(\mathbf{x}) e^{i\theta}$, where θ does not depend on $\mathbf{x}$.

(a) Prove that the corresponding Noether current is $j^\mu = i\phi^*(\partial^\mu \phi) - i\phi(\partial^\mu \phi)^*$. (2P)

(b) Now let us instead consider *local* $U(1)$ transformations $\phi(\mathbf{x}) \to \tilde{\phi}(\mathbf{x}) = \phi(\mathbf{x}) e^{i\theta(\mathbf{x})}$, where $\theta(\mathbf{x})$ is now a function of $\mathbf{x}$. Confirm that the Lagrangian $\mathcal{L} = -(\partial_\mu \phi)^*(\partial^\mu \phi)$ is generally *not* invariant under such local transformations. (1P)

(c) Replace the partial derivatives in the Lagrangian by so-called *covariant derivatives*

$$\partial^\mu \to D^\mu = \partial^\mu - iA^\mu(\mathbf{x}),$$

where $A^\mu(\mathbf{x})$ is the electromagnetic potential. Find a transformation

$$A^\mu(\mathbf{x}) \ \to \ \tilde{A}^\mu(\mathbf{x}) = A^\mu(\mathbf{x}) + f^\mu(\mathbf{x})$$

such that for given $\theta(\mathbf{x})$ the Lagrangian $\mathcal{L} = -(D_\mu \phi)^*(D^\mu \phi)$ is invariant under local $U(1)$ transformations. How can we interpret the transformation of $A^\mu(\mathbf{x})$? (3P)

(d) Consider now the combined Lagrangian

$$\mathcal{L} \ = \ -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)^* (D^\mu \phi)$$

and determine the equations of motion for ϕ, ϕ^*, and A_ν in Lorenz gauge. (3P)

(e) Show that the Noether current of this Lagrangian with respect to local $U(1)$ transformations is now given by $J^\mu = i\phi^*(D^\mu \phi) - i\phi(D^\mu \phi)^*$, reducing the equation of motion for the vector potential to $\Box A^\mu = \mu_0 J^\mu$. (2P)

(f) Verify that the Noether current is conserved, that is, $\partial_\mu J^\mu = 0$. To this end evaluate $\partial_\mu J^\mu$ and insert the equations of motion obtained in (d). (1P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, June 11 at 12:00.