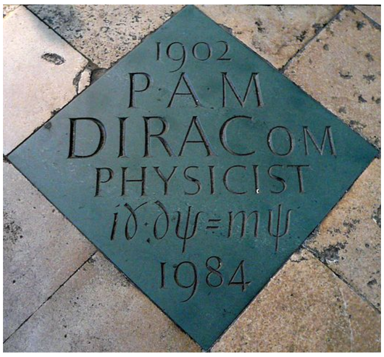
Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2021



Dirac memorial [Wikimedia].

Exercise 10.1: Helicity (4P)

The *helicity operator* is an operator (4×4-matrix) acting in spinor space. In the Dirac representation it is defined as the projection of the spin onto the direction of momentum

$$h(\vec{p}) = \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{p}\cdot\vec{\sigma} & \\ & \vec{p}\cdot\vec{\sigma} \end{pmatrix},$$

where $\vec{p}$ is the 3-momenteum and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T = (\sigma^x, \sigma^y, \sigma^z)^T$ is the vector operator consisting of Pauli matrices.

(a) Verify that $\big(h(\mathbf{p})\big)^2 = \mathbb{1}$. (1P)

(b) Show that for given momentum $\vec{p}$, the operators $P_\pm^{hel.}(\vec{p}) = \frac{1}{2}(\mathbb{1} \pm h(\vec{p}))$ is a complete set of orthogonal projection operators. (2P)

(c) Read about helicity and chirality in textbooks and explain qualitatively their difference. (1P)

Exercise 10.2: Covariance of the Dirac equation (8P)

According to the Lecture Notes, the 4-component Dirac wave function $\psi(\mathbf{x})$ transforms under $SO^+(3,1)$-transformations in the $(\alpha\beta)$-plane by the 'angle' $\theta_{(\alpha\beta)} \in \mathbb{R}$ as

$$\psi \to S\big(\theta_{(\alpha\beta)}\big)\psi \quad \text{where} \quad S\big(\theta_{(\alpha\beta)}\big) = \exp\Big(\frac{i}{2}\theta_{(\alpha\beta)}\sigma_{(\alpha\beta)}\Big) \quad \text{and} \quad \sigma_{(\alpha\beta)} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta].$$

(a) Convince yourself that in a representation of your choice the adjoint (= hermitean conjugate) of the γ-matrices is given by (1P)

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \quad \text{and} \quad {\gamma_\mu}^\dagger = \gamma_0\gamma_\mu\gamma_0$$

(b) Use (a) to show algebraically that the rotation generators $\sigma_{(12)}, \sigma_{(13)}, \sigma_{(23)}$ are hermitean while the boost generators $\sigma_{(01)}, \sigma_{(02)}, \sigma_{(03)}$ are anti-hermitean. (2P)

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(c) Show that

$$\psi^\dagger \psi = \left(\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*\right) \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \sum_{k=1}^{4} \psi_k^* \psi_k$$

is **not** invariant under Lorentz boosts. (1P)

(d) Prove that $S^\dagger_{(\alpha\beta)} = \gamma_0 \, S^{-1}_{(\alpha\beta)} \, \gamma_0$. *Hint:* Check that $\sigma^\dagger_{(\alpha\beta)} = \gamma_0 \, \sigma_{(\alpha\beta)} \, \gamma_0$. (2P)

(e) Show that instead $\bar{\psi}\psi$ is invariant under Lorentz boosts as well as under rotations, where $\bar{\psi} := \psi^\dagger \gamma^0$ is the so-called *adjoint spinor*. (1P)

(f) Show that $\bar{\psi}\gamma^\nu\psi$ transforms covariantly as a 4-vector.
Hint: The easiest way is to show that this condition is equivalent to $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$ which we have already proven in the Lecture Notes. (1P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, June 25 at 12:00.