SAMPLE SOLUTIONS EXERCISE 5

EXERCISE 5.1: POLYNOMIAL ACTION (3P)

In the lecture we derived the action of a point particle with mass m in a potential V:

$$S[\mathbf{x},\dot{\mathbf{x}}] = \int_{\tau_A}^{\tau_B} \Big(-mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + V(\mathbf{x})\Big)\,\mathrm{d}\tau.$$

Some theorists do not like this action because of the nasty square root. They would rather prefer a *polynomial* action. In this exercise let us study the following polynomial action

$$S[\mathbf{x},\dot{\mathbf{x}},\xi] = \int_{\tau_A}^{\tau_B} L(\mathbf{x},\dot{\mathbf{x}},\xi)\,\mathrm{d}\tau = \int_{\tau_A}^{\tau_B} \Big(\frac{1}{2\xi}\dot{x}_\mu\dot{x}^\mu - \frac{\xi m^2c^2}{2} + V(\mathbf{x})\Big)\,\mathrm{d}\tau,$$

where $\xi(\tau)$ is an additional independent function.

(a) Compute the variation $\delta S = 0$ with respect to δx^μ and $\delta\xi$ and derive the corresponding equations of motion. (2P)

(b) Insert the solution for $\xi(\tau)$ into the other equation of motion and show that we get the same results as for the ordinary action (see lecture notes). (1P)

SAMPLE SOLUTION

(a) The variation of the action reads

$$\delta S = \int \mathrm{d}\tau\Big(\frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta\dot{x}^\mu + \frac{\partial L}{\partial\xi}\delta\xi\Big).$$

The variation $\delta\xi(\tau)$ is independent while the variations δx^μ and $\delta\dot{x}^\mu$ are not independent. As usual, we get rid of $\delta\dot{x}^\mu$ via integration by parts: (1P)

$$\delta S = \int \mathrm{d}\tau\Big(\Big(\frac{\partial L}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial\dot{x}^\mu}\Big)\delta x^\mu + \frac{\partial L}{\partial\xi}\delta\xi\Big).$$

With the given Lagrangian

$$L(\mathbf{x},\dot{\mathbf{x}},\xi) = \frac{1}{2\xi}\dot{x}_\mu\dot{x}^\mu - \frac{\xi m^2c^2}{2} + V(\mathbf{x})$$

the equations of motion read: (1P)

$$\frac{\partial L}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial\dot{x}^\mu} = 0 \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\tau}p_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\dot{x}_\mu}{\xi} = \frac{\partial V}{\partial x^\mu}.$$

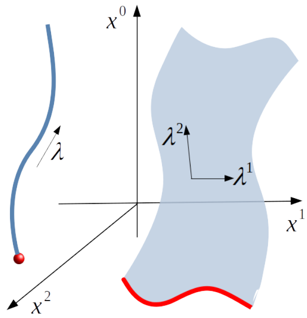
$$\frac{\partial L}{\partial\xi} = -\frac{1}{2\xi^2}\dot{x}_\mu\dot{x}^\mu - \frac{m^2}{2} = 0 \quad\Rightarrow\quad \xi = \frac{1}{mc}\sqrt{-\dot{x}^\mu\dot{x}_\mu}$$

(b) If we insert the solution of the second equation into the first one we get: (1P)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}p_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{mc\,\dot{x}_\mu}{\sqrt{-\dot{x}^\nu\dot{x}_\nu}} = \frac{\partial V}{\partial x^\mu}.$$

The generalized momentum and the equations of motion are identical with those of the usual action with the square root.

Exercise 5.2: String Theory **(9P)**



So far we have understood the physics of a relativistic particle. Such a particle is a zero-dimensional object which moves on a one-dimensional **world line** parameterized e.g. by $\lambda \mapsto \mathbf{x}(\lambda)$. For the action we have simply chosen the line integral over the relativistic length element $\mathrm{d}s = \sqrt{-\dot{x}_\mu\dot{x}^\mu}\,\mathrm{d}\lambda$.

Let us now do string theory. A string is a one-dimensional object which moves on a two-dimensional **world sheet** parameterized two parameters, e.g. by $\lambda^1, \lambda^2 \mapsto \mathbf{x}(\lambda^1, \lambda^2)$.

(a) As usual, the relativistic line element in the embedding space $\mathbb{R}_{3+1}$ is $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. On the world sheet the corresponding line element can be expressed as $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}\lambda^i\,\mathrm{d}\lambda^j$, where the Latin indices run over 1,2 and where the metric g_{ij} depends on the position on the world sheet. Give a general expression for g_{ij}. (1P)

(b) Prove: In the ordinary Euclidean $\mathbb{R}^n$ an infinitesimal area element $\mathrm{d}A$ spanned by two vectors $\mathrm{d}\vec{a}$ and $\mathrm{d}\vec{b}$ is given by $\mathrm{d}A = \sqrt{(\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{a})(\mathrm{d}\vec{b}\cdot\mathrm{d}\vec{b}) - (\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{b})^2}$. (1P)

(c) Guess an analogous formula for the area element spanned by $\mathrm{d}\mathbf{a}$ and $\mathrm{d}\mathbf{b}$ in the 3+1-dimensional Minkowski space. (1P)

(d) Let $\mathrm{d}\mathbf{a}$ and $\mathrm{d}\mathbf{b}$ be the displacements on the world sheet due to a variation of the parameters by $\mathrm{d}\lambda^1$ and $\mathrm{d}\lambda^2$, respectively. Show that the area element is given by $\mathrm{d}A = \sqrt{|g|}\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2$, where g is the determinant of g_{ij}. (1P)

(e) The **Nambu-Goto action** is defined as being proportional to the area of the sheet

$$S_{NG} = -T\int \mathrm{d}A$$

where $T > 0$ is a coupling constant (string tension). In a given parameterization of the world sheet, this action can be expressed as

$$S_{NG} = \int \mathcal{L}_{NG}\Big(\{\tfrac{\partial x^\mu}{\partial\lambda^1}\},\{\tfrac{\partial x^\nu}{\partial\lambda^2}\}\Big)\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2.$$

Determine the Lagrange density $\mathcal{L}_{NG}$. (1P)

(f) Analogous to the particle momentum $p_\mu = \frac{\partial L}{\partial\dot{x}^\mu}$, the string momentum is defined as

$$\Pi^i_\mu = \frac{\partial\mathcal{L}_{NG}}{\partial\left(\frac{\partial x^\mu}{\partial\lambda^i}\right)}.$$

As there is no potential, the equations of motion are then simply given by $\partial_i \Pi^i_\mu = 0$. Show that these classical equations of motion are equivalent to the analog of the wave equation $\Box x^\mu = 0$, where $\Box$ is the so-called **Laplace-Beltrami operator**: (2P)

$$\Box x^\mu = 0\,, \qquad \Box = \frac{1}{\sqrt{|g|}}\partial_i \sqrt{|g|}\, g^{ij}\partial_j$$

Hint: g^{ij} is the inverse matrix of g_{ij}.

(g) String theorist often prefer a different action, namely, the **Polyakov action**

$$S_P \;=\; -\frac{1}{4\pi\alpha}\int \mathrm{d}\lambda^1\,\mathrm{d}\lambda^2 \sqrt{|g|}\, g^{ij}\,\partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}.$$

This action is sometimes easier to handle since it is polynomial in the coordinates (no square root). Show that this action renders exactly the same equations of motion. (2P)

Sample Solution

(a) Simply apply the chain rule in the parameters

$$\begin{aligned}
\mathrm{d}s^2 &= \eta_{\mu\nu}\,\mathrm{d}x^\mu(\lambda^1,\lambda^2)\,\mathrm{d}x^\nu(\lambda^1,\lambda^2)\\
&= \eta_{\mu\nu}\left(\frac{\partial x^\mu}{\partial\lambda^i}\,\mathrm{d}\lambda^i\right)\left(\frac{\partial x^\nu}{\partial\lambda^j}\,\mathrm{d}\lambda^j\right)\\
&= \underbrace{\frac{\partial x_\mu}{\partial\lambda^i}\,\frac{\partial x^\mu}{\partial\lambda^j}}_{=g_{ij}(\lambda^1,\lambda^2)}\,\mathrm{d}\lambda^i\,\mathrm{d}\lambda^j
\end{aligned}$$

$$\Rightarrow \quad \boxed{g_{ij} \;=\; \frac{\partial x_\mu}{\partial\lambda^i}\,\frac{\partial x^\mu}{\partial\lambda^j}.}$$

(b) The scalar product can be interpreted as $\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{b} = |\,\mathrm{d}\vec{a}|\,|\,\mathrm{d}\vec{b}|\,\cos\phi$, where ϕ is the angle between the vectors. This is valid in any dimension. The infinitesimal area, on the other hand, is $\mathrm{d}A = |\,\mathrm{d}\vec{a}|\,|\,\mathrm{d}\vec{b}|\,\sin\phi$. Using $\sin^2\phi = 1-\cos^2\phi$ we get

$$\begin{aligned}
\mathrm{d}A^2 &= (\,\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{a})\,(\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{b})\sin^2\phi\\
&= (\,\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{a})\,(\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{b}) \;-\; (\,\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{a})\,(\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{b})\cos^2\phi\\
&= (\,\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{a})\,(\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{b}) \;-\; (\,\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{b})\,(\,\mathrm{d}\vec{x}\cdot\mathrm{d}\vec{b})
\end{aligned}$$

which proves the relation.

(c) Simply replace the Euclidean scalar product by the Minkowski scalar product:

$$\mathrm{d}A^2 \;=\; (\,\mathrm{d}a_\mu\,\mathrm{d}a^\mu)(\,\mathrm{d}b_\nu\,\mathrm{d}b^\nu) - (\,\mathrm{d}a_\mu\,\mathrm{d}b^\mu)(\,\mathrm{d}a_\nu\,\mathrm{d}b^\nu) \;=\; \mathrm{d}\mathbf{a}^2\,\mathrm{d}\mathbf{b}^2 - (\,\mathrm{d}\mathbf{a}\cdot\mathrm{d}\mathbf{b})^2$$

(d) Again we apply the chain rule

$$\begin{aligned} \mathrm{d}A^2 &= (\,\mathrm{d}a_\mu\,\mathrm{d}a^\mu)(\,\mathrm{d}b_\nu\,\mathrm{d}b^\nu) - (\,\mathrm{d}a_\mu\,\mathrm{d}b^\mu)(\,\mathrm{d}a_\nu\,\mathrm{d}b^\nu) \\ &= \Big(\frac{\partial x_\mu}{\partial\lambda^1}\,\mathrm{d}\lambda^1\frac{\partial x^\mu}{\partial\lambda^1}\,\mathrm{d}\lambda^1\Big)\Big(\frac{\partial x_\nu}{\partial\lambda^2}\,\mathrm{d}\lambda^2\frac{\partial x^\nu}{\partial\lambda^2}\,\mathrm{d}\lambda^2\Big) \\ &\quad - \Big(\frac{\partial x_\mu}{\partial\lambda^1}\,\mathrm{d}\lambda^1\frac{\partial x^\mu}{\partial\lambda^2}\,\mathrm{d}\lambda^2\Big)\Big(\frac{\partial x_\nu}{\partial\lambda^1}\,\mathrm{d}\lambda^1\frac{\partial x^\nu}{\partial\lambda^2}\,\mathrm{d}\lambda^2\Big) \\ &= \Big(\frac{\partial x_\mu}{\partial\lambda^1}\frac{\partial x^\mu}{\partial\lambda^1}\frac{\partial x_\nu}{\partial\lambda^2}\frac{\partial x^\nu}{\partial\lambda^2} - \frac{\partial x_\mu}{\partial\lambda^1}\frac{\partial x^\mu}{\partial\lambda^2}\frac{\partial x_\nu}{\partial\lambda^1}\frac{\partial x^\nu}{\partial\lambda^2}\Big)(\,\mathrm{d}\lambda^1)^2(\,\mathrm{d}\lambda^2)^2 \\ &= \Big(g_{11}g_{22} - g_{12}g_{21}\Big)(\,\mathrm{d}\lambda^1)^2(\,\mathrm{d}\lambda^2)^2 \;=\; g\,(\,\mathrm{d}\lambda^1 d\lambda^2)^2. \end{aligned}$$

Taking the square root we have to pay attention that the determinant may be negative (if one of the vectors is timelike) but we want to have a positive $\mathrm{d}A$, so we take

$$\boxed{\mathrm{d}A \;=\; \sqrt{|g|}\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2.}$$

(e) Comparing (d) with $S_{NG} = -T\int \mathrm{d}A = \int \mathcal{L}_{NG}\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2$ we get $\mathcal{L}_{NG} = -T\sqrt{|g|}$. Fully written out in components this reads:

$$S_{NG} \;=\; -T\int\sqrt{\left|\frac{\partial x_\mu}{\partial\lambda^1}\frac{\partial x^\mu}{\partial\lambda^1}\frac{\partial x_\nu}{\partial\lambda^2}\frac{\partial x^\nu}{\partial\lambda^2} - \frac{\partial x_\mu}{\partial\lambda^1}\frac{\partial x^\mu}{\partial\lambda^2}\frac{\partial x_\nu}{\partial\lambda^1}\frac{\partial x^\nu}{\partial\lambda^2}\right|}\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2$$

(f) We first compute the world sheet momenta Π^1_μ and Π^2_μ: (1P)

$$\begin{aligned} \Pi^1_\mu &= \frac{\partial\mathcal{L}_{NG}}{\partial\left(\frac{\partial x^\mu}{\partial\lambda^1}\right)} = 2\frac{-T}{2\sqrt{|g|}}\Big(\frac{\partial x_\nu}{\partial\lambda^2}\frac{\partial x^\nu}{\partial\lambda^2}\frac{\partial x_\mu}{\partial\lambda^1} - \frac{\partial x_\nu}{\partial\lambda^1}\frac{\partial x^\nu}{\partial\lambda^2}\frac{\partial x_\mu}{\partial\lambda^2}\Big) \\ &= -\frac{T}{\sqrt{|g|}}\,(g_{22}\partial_1 - g_{12}\partial_2)x_\mu \end{aligned}$$

and likewise

$$\Pi^2_\mu \;=\; -\frac{T}{\sqrt{|g|}}\,(g_{11}\partial_2 - g_{21}\partial_1)x_\mu\,.$$

In order to bring this into contact with the Laplace-Beltrami operator, we need the metric tensor with upper indices. According to the hint in the exercise, g_{ij} is the inverse of g_{ij}. Fortunately, the inversion of a 2×2 matrix is trivial:

$$\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix} = \frac{1}{g}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{pmatrix}$$

The world sheet momenta can then be rewritten as (1P)

$$\Pi^1_\mu \;=\; -T\sqrt{|g|}(g^{11}\partial_2 + g^{12}\partial_2)x_\mu\,,\qquad \Pi^2_\mu \;=\; -T\sqrt{|g|}(g^{22}\partial_2 + g^{21}\partial_1)x_\mu$$

which can now neatly be combined in a single expression:

$$\boxed{\Pi^i_\mu \;=\; -T\sqrt{|g|}\,g^{ij}\,\partial_j\,x_\mu\,.}$$

Now, if we multiply the equations of motion $\partial_i\Pi^i_\mu = 0$ from the left with $-\frac{1}{T\sqrt{|g|}}$, we arrive at

$$\frac{1}{\sqrt{|g|}}\,\partial_i\,\sqrt{|g|}\,g^{ij}\,\partial_j\,x_\mu \;=\; 0$$

(g) For the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha}\int \mathrm{d}\lambda^1\,\mathrm{d}\lambda^2 \sqrt{|g|}\, g^{ij}\partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}$$

the corresponding Lagrange density

$$S_P = \int \mathcal{L}_P\Big(\{\tfrac{\partial x^\mu}{\partial \lambda^1}\}, \{\tfrac{\partial x^\nu}{\partial \lambda^2}\}\Big)\, \mathrm{d}\lambda^1\,\mathrm{d}\lambda^2.$$

is given by

$$\mathcal{L}_P = -\frac{1}{4\pi\alpha}\sqrt{|g|}\, g^{ij}\,\partial_i x^\mu \partial_j x^\nu \eta_{\mu\nu}$$

Then the momenta read

$$\Pi^i_\mu = \frac{\partial \mathcal{L}_P}{\partial\left(\frac{\partial x^\mu}{\partial \lambda^i}\right)} = \frac{\partial \mathcal{L}_P}{\partial\left(\partial_i x^\mu\right)} = -\frac{1}{2\pi\alpha}\sqrt{|g|}\, g^{ij}\,\partial_j x^\nu \eta_{\mu\nu} = -\frac{1}{2\pi\alpha}\sqrt{|g|}\, g^{ij}\,\partial_j x_\mu$$

Identifying $T = \frac{1}{2\pi\alpha}$ this is exactly the same momentum as in (f), hence the equations of motion $\partial_i \Pi^i_\mu = 0$ will be the same as well.

(Σ = 12P)