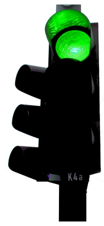
SAMPLE SOLUTIONS EXERCISE 4

EXERCISE 4.1: ILLEGAL CAR RACE (4P)

A car driver, who crossed a red traffic light during an illegal car race in Tübingen, tells the police that it was actually green due to relativistic effects.

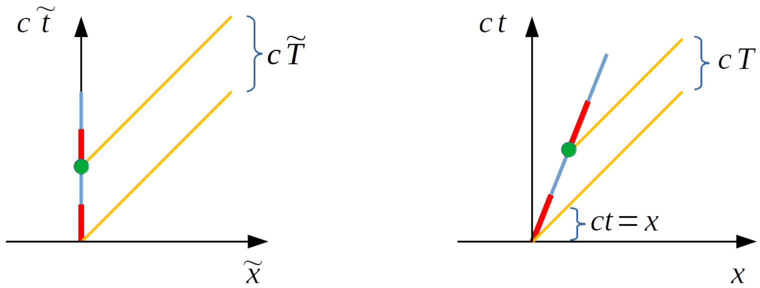


(a) Derive the formula for the relativistic Doppler effect. (2P)

(b) How fast was the car driving? ($\lambda_{\text{red}} = 660\text{nm}$, $\lambda_{\text{green}} = 540\text{nm}$) (1P)

(c) The rest mass of the car is 2000 kg. How large was the (relativistic) kinetic energy? Compare this energy with the total energy consumed in Germany in 2018. (`https://de.wikipedia.org/wiki/Energieverbrauch`) (1P)

SAMPLE SOLUTION



(a) The figure shown above sketches how the relativistic Doppler effect can be derived. Consider an oscillator resting in system $\tilde{S}$ (left panel). It emits wave fronts with the speed of light, their time difference being $\tilde{T}$. The right panel shows the same situation in the laboratory frame, in which the oscillator is moving to the right with velocity c. The Lorentz transformation reads (1P)

$$x = \gamma(\tilde{x} + \beta c\tilde{t}), \qquad ct = \gamma(c\tilde{T} + \beta\tilde{x})$$

Hence the event marked by the green bullet, which has the coordinates $(0, c\tilde{T})$ in the system $\tilde{S}$, has now the coordinates $(\gamma\beta c\tilde{T}, \gamma c\tilde{T})$. The time span (vertical distance) between the emitted light rays is

$$cT = \gamma c\tilde{T} - \gamma\beta c\tilde{T} \quad \Rightarrow \quad T = \gamma(1-\beta)\tilde{T} = \sqrt{\frac{1-\beta}{1+\beta}}\,\tilde{T}$$

The frequencies transform inversely, i.e.: (1P)

$$\nu_{\text{blue-shifted}} = \sqrt{\frac{1+\beta}{1-\beta}}\,\nu_{\text{emitted}}.$$

(b) We have to solve: (1P)

$$R = \frac{\lambda_{\text{red}}}{\lambda_{\text{green}}} = \frac{660}{540}\,\frac{\nu_{\text{green}}}{\nu_{\text{red}}} = \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\Rightarrow \qquad \beta = \frac{R^2-1}{R^2+1} \approx 0.198 \quad \Rightarrow \quad v \approx 213.000.000 \text{ km/h}$$

(c) The relativistic kinetic energy is $E_{\text{kin}} = m_0 c^2(\gamma - 1)$. This gives (1P)

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{R + R^{-1}}{2} \approx 1.0202.$$

$$\Rightarrow E_{\text{kin}} \approx 2000 \text{ kg } 9 \times 10^{16} \frac{\text{m}^2}{\text{s}^2} \times 0.0202 \approx 3.63 \times 10^{18} \text{J}$$

According to Wikipedia, the total energy consumed in Germany in 2017 was 1.31×10^{19} J. The kinetic energy of the car would be roughly one quarter of this amount.

Exercise 4.2: Isometries in 2+1 dimensions (5P)

In 2+1 dimensions the generators of the Minkowski isometries are given by

$$\lambda_{(01)}{}^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix}, \quad \lambda_{(02)}{}^{\mu}{}_{\nu} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix}, \quad \lambda_{(12)}{}^{\mu}{}_{\nu} = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix}$$

(a) Compute the corresponding transformations $\Lambda_{(\alpha\beta)}(\theta) = \exp\big(\theta \lambda_{(\alpha\beta)}\big)$ and discuss briefly their interpretation. (1P)

(b) Confirm that the transformations computed in (a) are isometries with respect to the Minkowski metric (it suffices to check $\Lambda_{(01)}$). (1P)

(c) Prove that the integral

$$I = \int_{\mathbb{R}_{2+1}} f(\mathbf{x} \cdot \mathbf{x}) := \iiint \mathrm{d}x^0 \, \mathrm{d}x^1 \, \mathrm{d}x^2 \, f(x^\mu x_\mu)$$

is a scalar, that is, it is invariant under the isometries computed in (a). (1P)

(d) The product (subsequent application) of two isometries is again an isometry. As an example let us consider the isometry $\Lambda^*(\theta) := \Lambda_{(01)}(\theta)\Lambda_{(02)}(\theta)$. Find a straight line (analogous to a rotation axis) in the 2+1-dimensional Minkowski space which is invariant under $\Lambda^*(\theta)$. (2P)

Sample Solution

(a) The computation of the matrix exponential function follows the same line as in the previous exercise 2.2. These transfomations can be interpreted as follows: One has two Lorentz boosts in x and y (or x^1 and x^2) direction (1P)

$$\Lambda_{(01)}{}^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\theta_{(01)} & \sinh\theta_{(01)} & \\ \sinh\theta_{(01)} & \cosh\theta_{(01)} & \\ & & 1 \end{pmatrix}, \quad \Lambda_{(02)}{}^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\theta_{(02)} & & \sinh\theta_{(02)} \\ & 1 & \\ \sinh\theta_{(02)} & & \cosh\theta_{(02)} \end{pmatrix}.$$

as well as one ordinary rotation in the xy-plane

$$\Lambda_{(12)}{}^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & \\ & \cos\theta_{(12)} & -\sin\theta_{(12)} \\ & \sin\theta_{(12)} & \cos\theta_{(12)} \end{pmatrix}$$

(b) The isometry condition reads

$$\eta_{\mu\nu} = \eta_{\rho\tau}\Lambda^{\rho}{}_{\mu}\Lambda^{\tau}{}_{\nu}.$$

where $\eta = \mathrm{diag}(-1,1,1)$. Inserting $\Lambda_{(01)}$ we get for the right hand side (1P)

$$\eta_{\rho\tau}{\Lambda_{(01)}}^{\rho}{}_{\mu}{\Lambda_{(01)}}^{\tau}{}_{\nu} = \begin{pmatrix} -\cosh^2\theta_{(01)} + \sinh^2\theta_{(01)} & 0 & 0 \\ & \cosh^2\theta_{(01)} - \sinh^2\theta_{(01)} & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\mu\nu}.$$

Since $\cosh^2 x - \sinh^2 x = 1$ we arrive at the desired result.

Correction advice: Requires calculation involving $\cosh^2 x - \sinh^2 x = 1$

(c) Under the transformation $x^\mu \mapsto \tilde{x}^\mu = \Lambda^{\mu}{}_{\nu}x^\nu$ the argument of the function is a scalar, that is $x^\mu x_\mu = \tilde{x}^\mu x_\mu$. Therefore, the integral turns into

$$\iiint \mathrm{d}x^0\,\mathrm{d}x^1\,\mathrm{d}x^2\, f(x^\mu x_\mu) \mapsto \iiint \mathrm{d}\tilde{x}^0\,\mathrm{d}\tilde{x}^1\,\mathrm{d}\tilde{x}^2\, J\, f(\tilde{x}^\mu \tilde{x}_\mu).$$

where J is the Jacobi determinant. The Jacobi matrix is Λ itself, i.e., $J = \det(\Lambda)$. All what we have to check is that the determinant of matrices Λ in (a) equals 1. This is indeed the case (again thanks to the relation $\cosh^2 x - \sinh^2 x = 1$).

Correction advice: The essential point here is the awareness and calculation of the Jacobi determinant. (1P)

(d) First we compute (1P)

$$\Lambda^*(\theta) = \begin{pmatrix} \cosh^2(\theta) & \sinh(\theta) & \sinh(\theta)\cosh(\theta) \\ \sinh(\theta)\cosh(\theta) & \cosh(\theta) & \sinh^2(\theta) \\ \sinh(\theta) & 0 & \cosh(\theta) \end{pmatrix}$$

As Λ^* maps the origin to the origin, the questioned straight line has to run trough the origin. Therefore, invariant lines are characterized by the eigenvectors of Λ^*. Using *Mathematica®* we find that there is an eigenvalue 1 corresponding to the eigenvector (1P)

$$x^0 = \frac{1}{\sinh\theta} - \coth\theta\,, \quad x^1 = -1\,, \quad x^2 = 1.$$

The straight line along this vector is invariant under the isometry. In may be written in a parameterized form as

$$\vec{x} = \begin{pmatrix} \kappa(\frac{1}{\sinh\theta} - \coth\theta) \\ -\kappa \\ \kappa \end{pmatrix}, \qquad \kappa \in \mathbb{R} \tag{1}$$

Exercise 4.3: Relativistic Acceleration **(3P)**

The 4-velocity and the 4-acceleration are defined by $\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}$ and $\mathbf{a} = \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathbf{x}$, where τ denotes the proper time and $\mathbf{x} = \{x^\mu\} = \begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$ is the position 4-vector.

(a) Show that the 4-acceleration is orthogonal on the 4-velocity, i.e., $\mathbf{a} \cdot \mathbf{u} = 0$. (1P)

(b) Prove that

$$\mathbf{a} = \{a^\mu\} = \gamma^4 \begin{pmatrix} \frac{1}{c}\vec{a}\cdot\vec{v} \\ \vec{a} + \frac{1}{c^2}\vec{v}\times(\vec{v}\times\vec{a}) \end{pmatrix},$$

where $\vec{v}$ is the ordinary 3-velocity, $\gamma = (1 - v^2/c^2)^{-1/2}$, and $\vec{a} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{v}$. (2P)

Sample Solution

(a) The 4-velocity in proper time parameterization is has the constant length

$$||\mathbf{u}||^2 \;=\; \eta_{\mu\nu}u^\mu u^\nu = -c^2.$$

Therefore, its derivative with respect to the proper time τ vanishes:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\eta_{\mu\nu}u^\mu u^\nu = \eta_{\mu\nu}\Big(\dot{u}^\mu u^\nu + u^\mu \dot{u}^\nu\Big) = 2\eta_{\mu\nu}a^\mu u^\nu = 2a_\mu u^\mu = 2\mathbf{a}\cdot\mathbf{u}.$$

Consequently $\mathbf{a} \cdot \mathbf{u} = 0$.

(b) The trajectory can be parameterized either by the proper time τ or by the coordinate time t. The corresponding derivatives are related by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}}{\mathrm{d}t}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Therefore, the 4-velocity is given by (cf. lecture notes)

$$\mathbf{u} = \{u^\mu\} = \begin{pmatrix} \gamma c \\ \gamma\vec{v} \end{pmatrix}.$$

To compute the 4-acceleration, we have to differentiate this again with respect to τ, keeping in mind that the time-dependent pre-factor γ in $\mathbf{u}$ needs to be differentiated as well. Noting that (1P)

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} \;=\; \frac{\gamma^3}{c^2}\vec{v}\cdot\vec{a}.$$

a straight-forward calculation gives

$$\mathbf{a} = \{a^\mu\} = \begin{pmatrix} \gamma\frac{\mathrm{d}}{\mathrm{d}t}\gamma c \\ \gamma\frac{\mathrm{d}}{\mathrm{d}t}\gamma\vec{v} \end{pmatrix} = \begin{pmatrix} c\gamma(\frac{\mathrm{d}}{\mathrm{d}t}\gamma) \\ \gamma^2\vec{a} + \gamma(\frac{\mathrm{d}}{\mathrm{d}t}\gamma)\vec{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\gamma^4\vec{v}\cdot\vec{a} \\ \gamma^2\vec{a} + \frac{1}{c^2}\gamma^4(\vec{v}\cdot\vec{a})\vec{v} \end{pmatrix}$$

The second component can be rewritten as $\gamma^2\vec{a} + \frac{1}{c^2}\gamma^4(\vec{v}\cdot\vec{a})\vec{v} = \gamma^4\Big(\vec{a} + \frac{1}{c^2}\vec{v}\times(\vec{v}\times\vec{a})\Big)$. To see that we apply the famous BAC-CAB formula to the double cross product on the right hand side:

$$\gamma^4\Big(\vec{a} + \frac{1}{c^2}\vec{v}\times(\vec{v}\times\vec{a})\Big) \;=\; \gamma^4\Big(\vec{a} + \frac{\vec{v}(\vec{v}\cdot\vec{a})}{c^2} - \frac{\vec{a}(\vec{v}\cdot\vec{v})}{c^2}\Big)$$
$$= \gamma^4\Big(1 - \frac{v^2}{c^2}\Big)\vec{a} + \gamma^4\frac{\vec{a}(\vec{v}\cdot\vec{v})}{c^2} \;=\; \gamma^2\vec{a} + \gamma^4\frac{(\vec{v}\cdot\vec{v})\vec{a}}{c^2}$$

This completes the proof. (1P)

(Σ = 12P)