Sample Solutions Exercise 3

Exercise 3.1: Ladder Paradox (6P)

A physicist is faced with the problem that his/her ladder does not fit into the garage because the rest length of the ladder exceeds the size of the garage by 25% (see figure). But apparently there is a simple relativistic solution: We would move the ladder into the garage at 80% of the velocity of light, and then we would instantly close the doors just in the moment when the Lorentz-contracted ladder fits into the garage. However, from the perspective of the ladder, the garage is Lorentz-contracted, so it seems to be impossible to close the doors.

(a) Let a be the size of the garage in its rest frame. Compute β and γ. Is the Lorentz-contracted length b of the ladder in fact smaller than the size of the garage? (1P)

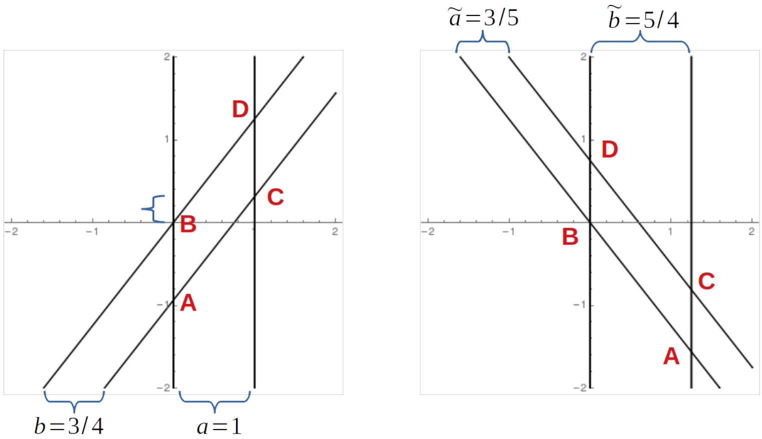
(b) Let us first keep both doors open and let the ladder pass freely. Draw the Minkowski diagrams in the frames of the garage and the ladder in exact proportions (maßstabsgetreu). (2P)
Analyze the sequence of the following four events:
A: Tip of ladder enters garage
B: End of ladder enters garage
C: Tip of ladder leaves garage
D: End of the ladder leaves garage. (1P)

(c) The physicist in the rest frame of the garage says: "In the right moment I can close both doors". An observer co-moving with the ladder says: "The ladder is too long, therefore it is in principle impossible to close both doors, even at different times". Who is right, who is wrong, and why? (2P)

Sample Solution

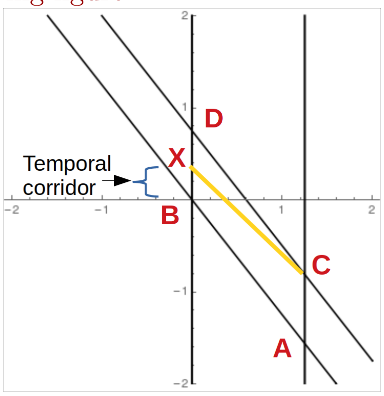
(a) Since $\beta = 0.8 = 4/5$ we have $\gamma = 1/\sqrt{1-\beta^2} = 5/3$. If a is the size of the garage, the length of the ladder is $\tilde{b} = \frac{5}{4}a$ while the Lorentz-contracted length of the ladder is $b = \tilde{b}/\gamma = \frac{3}{4}a$. Answer: Yes, the contracted ladder is indeed smaller.

(b) The Minkowski diagrams look like this: Left: Frame of the garage. (1P)
Right: Co-moving frame of the ladder. (1P)



The paradox is related to the sequence of events: In the rest frame of the garage, B happens before C. In the reference frame of the ladder, the temporal sequence of these events is exchanged. (1P)

(c) The resolution of the paradox is as follows: Suppose that the physicist keeps the exit door closed. Then the ladder will collide with the exit door. This will cause a physical impact (a negative acceleration) that will modify the ladder in a non-predictable manner, but we can be sure that the influence of the impact can propagate to the left not faster than the velocity of light. That is, upon collision in C, the tip of the ladder will come to a halt while the end of the ladder will continue to move for a while before it decelerates. We marked the segment of the light cone, below which the impact has no influence, by a yellow line in the following figure.



In the rest frame of the ladder, this means that although the top of the ladder has already collided with the exit door in point C, the end of the ladder still propagates without a change for some while, at least until the point X. As can be seen in the figure, C happens after B in the co-moving frame, leaving the possibility to close the entrance door in the temporal window between B and X.

To conclude, the answer of the physicist is correct: He/she can close both doors in any frame. Of course, upon deceleration, the ladder expands again and eventually it will probably not fit into the garage (and probably the whole garage will explode), but this is a different story.

EXERCISE 3.2: DISTANCE-DEPENDENT SOLUTION OF THE WAVE EQUATION (3P)

In this exercise we want to outline an interesting way finding solutions of the wave equations which depend exclusively on the relativistic distance $\mathbf{x}^2 = x_\mu x^\mu$. These solutions are then invariant under Lorentz transformations.

(a) Consider the $d+1$-dimensional Minkowski space equipped with the metric $g_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$. Let $s = |\mathbf{x}| = \sqrt{x_\mu x^\mu}$ be the relativistic distance. Show that for a function $g(s)$ which depends only on s, the partial differential equation

$$\Box g = \partial_\mu \partial^\mu g(s) = 0 \qquad\qquad s \neq 0$$

reduces to an autonomous[1] ordinary differential equation for $g(s)$ in terms of s. To find this ODE, consider first the 1+1-dimensional case, then the 2+1-dimensional case and finally guess the form of the ODE for general $d \in \mathbb{N}$. (2P)

(b) Solve the ODE for general d obtained in (a). Note the special case $d = 1$. (1P)

SAMPLE SOLUTION

(a) First we consider the 1+1-dimensional case $(x^0, x^1) = (t, x)$:

$$\Box g(s) = (\partial_x^2 - \partial_t^2)\, g\big(\sqrt{x^2 - t^2}\big)$$

$$\Rightarrow \quad \frac{g'(s)}{s} + g''(s) = 0\,,$$

which is in fact an autonomous ODE. Similarly, we get the ODE in 2+1 dimensions, where $(x^0, x^1, x^2) = (t, x, y)$: (1P)

$$\Box g(s) = (\partial_x^2 + \partial_y^2 - \partial_t^2)\, g\big(\sqrt{x^2 + y^2 - t^2}\big)$$

$$\Rightarrow \quad \frac{2g'(s)}{s} + g''(s) = 0\,,$$

This can be repeated in higher dimensions, leading to an increasing pre-factor in the first term. This allows us to conjecture the general result (1P)

$$\frac{d\, g'(s)}{s} + g''(s) = 0$$

(b) Solving this differential equation (e.g. with *Mathematica®*) yields a power law:

$$g(s) = A + Bs^{1-d}\,,$$

were A and B are integration constants. The only exception is $d = 1$. Here the ODE has the solution (1P)

$$g(s) = A + B\, \ln s\,.$$

[1] Autonomous means that only the variable s and derivatives with respect to s occur, while the coordinates x^μ do no longer appear individually.

Exercise 3.3: Fourier transformations in Minkowski space (3P)

In the $(d+1)$-dimensional Minkowski space, the Fourier transformation $f(\mathbf{x}) \mapsto \tilde{f}(\mathbf{k})$ is defined by

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}^{d+1}x \; e^{ik_\mu x^\mu} \, f(\mathbf{x})\,.$$

(a) Write down the inverse Fourier transformation. (1P)

(b) The Greens function of the Klein-Gordon equation is defined by

$$(\Box - m^2)G(\mathbf{x}) = \delta^{d+1}(\mathbf{x}).$$

Calculate the Fourier transformation of this equation and find a solution for the transformed Greens function $\tilde{G}(\mathbf{k})$. (1P)

(c) Prove the following statement: If a given function $f(\mathbf{x})$ is Lorentz-invariant under $\mathbf{x} \to \Lambda\mathbf{x}$, then the Fourier-transformed function $\tilde{f}(\mathbf{k})$ will also be Lorentz-invariant under $\mathbf{k} \to \Lambda\mathbf{k}$. (1P)

Sample Solution

(a) The inverse Fourier transformation reads

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}^{d+1}k \, e^{-ik_\mu x^\mu} \, \tilde{f}(\mathbf{k})\,.$$

(b) We simply insert the inverse transformation from (a):

$$\begin{aligned}
(\Box - m^2)G(\mathbf{x}) &= \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}^{d+1}k \, (\underbrace{\partial_\mu \partial^\mu}_{=-k_\mu k^\mu} - m^2) e^{-ik_\mu x^\mu} \, \tilde{G}(\mathbf{k}) \\
&= \frac{1}{(2\pi)^{d+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}^{d+1}k e^{-ik_\mu x^\mu} \, (-\mathbf{k}^2 - m^2)\tilde{G}(\mathbf{k}) = \delta^{d+1}(\mathbf{x}).
\end{aligned}$$

Then we apply an forward transformation to both sides, giving

$$(-\mathbf{k}^2 - m^2)\tilde{G}(\mathbf{k}) = 1 \quad \Rightarrow \quad \boxed{\tilde{G}(\mathbf{k}) = \frac{1}{-\mathbf{k}^2 - m^2}\,.}$$

Remark: In other textbooks one often finds $G(k) = (\mathbf{k}^2 - m^2)^{-1}$. The minus sign here is a consequence of the 'mostly plus' convention.

(c) The key point is that we have a scalar product in the exponential of the Fourier transformation. For a scalar product we have $g(\Lambda\mathbf{a}, \mathbf{b}) = g(\mathbf{a}, \Lambda^{-1}\mathbf{b})$, where Λ^{-1} is the inverse Lorentz transformation.

$$\tilde{f}(\Lambda\mathbf{k}) = \iint \mathrm{d}^{d+1}x \, e^{ig(\Lambda\mathbf{k},\mathbf{x})} \, f(\mathbf{x}) = \iint \mathrm{d}^{d+1}x \, e^{ig(\mathbf{k},\Lambda^{-1}\mathbf{x})} \, f(\mathbf{x})$$

Now we make a variable transformation $\mathbf{x} \to \Lambda\mathbf{x}$ in the integrand:

$$= \iint \mathrm{d}^{d+1}x \, e^{ig(\mathbf{k},\overbrace{\Lambda^{-1}\Lambda}^{=1}\mathbf{x})} \, \underbrace{f(\Lambda\mathbf{x})}_{=f(\mathbf{x})} = \tilde{f}(\mathbf{k})$$

Correction advice: This can be proven in various different ways. The key point is the insight that the exponential is a scalar, $k_\mu x^\mu$ does not change if both $\mathbf{x}$ and $\mathbf{k}$ are transformed with the same Lorentz transformation. In this way we can pass on the Lorentz transformation from $\mathbf{x}$ to $\mathbf{k}$.

$(\Sigma = \mathbf{12P})$