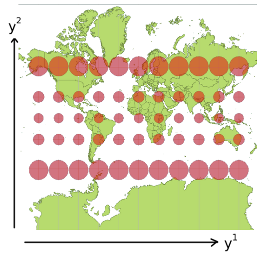
SAMPLE SOLUTIONS EXERCISE 2

EXERCISE 2.1: MERCATOR PROJECTION (3P)

Nautical maps use the Mercator projection. The Mercator projection maps the coordinates of the sphere (the longitude $x^1 = \phi \in [0, 2\pi]$ and the latitude $x^2 = \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ measured from the equator) onto new coordinates y^1, y^2 on the map. The longitude is mapped identically by $y^1 \equiv x^1$, while the latitude is stretched by a certain function $y^2 := f(x^2)$. This function is chosen in such a way that the map preserves angles and proportions, but not the area (see figure). These properties manifests themselves in the fact that the matrix of the metric tensor on the map is proportional to the identity.



(a) Construct the metric tensor $g_{\mu\nu}(\phi, \theta) = \begin{pmatrix} g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\theta\theta} \end{pmatrix}$ on the sphere. (1P)

(b) Compute the metric tensor $\tilde{g}_{ij}(y^1, y^2)$ on the map for general f. (1P)

(c) Determine the function f in such a way that the map is angular-preserving. (1P)

SAMPLE SOLUTION

(a) If we vary the latitude θ by $\mathrm{d}\theta$ this corresponds to a length element $\mathrm{d}l = \mathrm{d}\theta$ on the surface of the unit sphere. Contrarily, if we vary ϕ by $\mathrm{d}\phi$ the length element depends on θ: it is zero at the poles and maximal at the equator. More specifically, it is easy to see that $\mathrm{d}l = \cos(\theta)\,\mathrm{d}\phi$. Both changes are locally perpendicular, hence the line element is:

$$\mathrm{d}l^2 = \cos^2(\theta)\,\mathrm{d}\phi^2 + \mathrm{d}\theta^2$$

Consequently the metric tensor on the unit sphere is (1P)

$$g_{\mu\nu} = \begin{pmatrix} g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\theta\theta} \end{pmatrix} = \begin{pmatrix} \cos^2\theta & \\ & 1 \end{pmatrix}$$

Correction advice: Some plausible arguments should be given for the construction. Do not accept $g_{\phi\phi}(\phi, \theta) = \sin^2(\theta)$ because θ is measured from the equator.

(b) From the invariance condition

$$\mathrm{d}l^2 = g_{\mu\nu}(\mathbf{x})\,\mathrm{d}x^\mu x^\nu = \tilde{g}_{ij}(\mathbf{y})\,\mathrm{d}y^i\,\mathrm{d}y^j$$

one can 'derive' the transformation formula

$$g_{\mu\nu}(\mathbf{x}) = \frac{\partial y^i}{\partial x^\mu}\frac{\partial y^j}{\partial x^\nu}\tilde{g}_{ij}(y)\,.$$

Only the partial derivatives $\frac{\partial y^0}{\partial x^0} = 1$ and $\frac{\partial y^0}{\partial x^0} = f'(x^1)$ contribute, hence

$$\tilde{g}_{00} = \cos^2\theta\,, \quad \tilde{g}_{01} = \tilde{g}_{10} = 0\,, \quad \tilde{g}_{11} = \frac{1}{\big(f'(\theta)\big)^2} \quad \Rightarrow \quad \tilde{g}_{ij} = \begin{pmatrix} \cos^2\theta & \\ & [f'(\theta)]^{-2} \end{pmatrix}$$

(c) The condition that $\tilde{g}$ is proportional to the identity, i.e., $g_{11} = \tilde{g}_{11}$, leads to a first order differential equation

$$f'(\theta) = \frac{1}{\cos\theta}$$

which has the solution (using *Mathematica®*):

$$f(\theta) = 2\,\mathrm{arctanh}\Big[\tan\Big(\frac{\theta}{2}\Big)\Big] + C\,,$$

where the integration constant is zero if the equator is mapped onto the equator. This formula is equivalent to the one given in Wikipedia:

$$f(\theta) = \ln\Big[\tan\Big(\frac{\pi}{4} + \frac{\theta}{2}\Big)\Big]$$

Exercise 2.2: Simple Lie algebras and the exponential function (6P)

Consider an abstract operator λ obeying $\lambda^2 = -\mathbb{1}$, where $\mathbb{1} = \lambda^0$ is the identity.

(a) Write down the Taylor series of $\Lambda = \exp(\phi\lambda)$, where $\phi \in \mathbb{R}$. (1P)

(b) Separate the Taylor series into an even and an odd part in order to show that (2P)

$$\exp(\phi\lambda) = \mathbb{1}\cos\phi + \lambda\sin\phi.$$

(c) Apply the result from (b) to the representations $\lambda = i$ and $\lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (1P)

(d) Repeat (b) for $\lambda^2 = +\mathbb{1}$ and find a non-trivial 2×2 matrix representation. (2P)

Sample Solution

(a) The Taylor series reads (1P)

$$\Lambda = \exp(\phi\lambda) = \sum_{k=0}^{\infty} \frac{\phi^k \lambda^k}{k!}.$$

(b) The Taylor series can be separated into an even and an odd part (2P)

$$\begin{aligned}\exp(\phi\lambda) &= \sum_{n=0}^{\infty} \frac{\phi^{2n}\lambda^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\phi^{2n+1}\lambda^{2n+1}}{(2n+1)!} \\ &= \mathbb{1}\sum_{n=0}^{\infty} \frac{\phi^{2n}(-1)^n}{(2n)!} + \lambda\sum_{n=0}^{\infty} \frac{\phi^{2n+1}(-1)^n}{(2n+1)!} = \mathbb{1}\cos\phi + \lambda\sin\phi.\end{aligned}$$

(c) For the complex representation $\lambda = i$ we get the well-known formula

$$e^{i\phi} = \cos\phi + i\sin\phi$$

which is a phase factor describing a rotation in the complex plane. For the matrix representation we get:

$$\exp\left[\phi\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cos\phi + \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sin\phi = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}$$

which is just a rotation matrix in $\mathbb{R}^2$. (1P)

(d) For $\lambda^2 = +\mathbb{1}$ we only have to delete the alternating signs $(-1)^n$, i.e.: (1P)

$$\exp(\phi\lambda) = \mathbb{1}\sum_{n=0}^{\infty}\frac{\phi^{2n}}{(2n)!} + \lambda\sum_{n=0}^{\infty}\frac{\phi^{2n+1}}{(2n+1)!} = \mathbb{1}\cosh\phi + \lambda\sinh\phi.$$

A possible matrix representation is (1P)

$$\exp\left[\phi\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cosh\phi + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\sinh\phi = \begin{pmatrix}\cosh\phi & \sinh\phi\\ \sinh\phi & \cosh\phi\end{pmatrix}$$

In principle we could use any of the three Pauli matrices.

⇒ PLEASE TURN OVER

Exercise 2.3: Operator exponential function (3P)

In the last lecture we studied the translation operator $\exp(a\partial_x)$. Let us now consider the operator $\exp(bx\partial_x)$ acting on functions $f : \mathbb{R} \to \mathbb{R}$ with a real parameter $b > 0$.

(a) Compute $\exp(bx\partial_x)f(x)$. (*Hint:* Consider $\exp(bx\partial_x) = [\exp(\frac{b}{N}x\partial x)]^N$ for $N \to \infty$). What kind of transformation does this operator describe? (2P)

(b) Determine the eigenfunctions and eigenvalues of the generator $x\partial_x$, that is, find functions $\phi : \mathbb{R} \to \mathbb{R}$ and eigenvalues $\lambda \in \mathbb{R}$ such that $x\partial_x\phi(x) = \lambda\phi(x)$. Use the result to compute $\exp(bx\partial_x)\phi(x)$. (1P)

Sample Solution

(a) The relation

$$\exp(bx\partial_x)f(x) \;=\; \left[\exp(\tfrac{b}{N}x\partial x)\right]^N f(x)$$

holds for any N. Taking $N \to \infty$, the prefactor $\frac{b}{N}$ becomes infinitesimally small so that we can expand the exponential function to first order:

$$\exp(bx\partial_x)f(x) \;=\; \lim_{N\to\infty}\left[1 + \tfrac{b}{N}x\partial x)\right]^N f(x)$$

If we compare $\left[1 + \frac{bx}{N}\partial_x)\right]$ with the Taylor expansion $f(x+\epsilon) \approx f(x) + \epsilon f'(x)$ we can identify $\epsilon \equiv \frac{bx}{N}$, hence

$$\left[1 + \tfrac{b}{N}x\partial_x)\right]f(x) \;=\; f(x) + \tfrac{bx}{N}f'(x) \;\approx\; f\big(x + \tfrac{bx}{N}\big) \;=\; f\big((1+\tfrac{b}{N})x\big).$$

$$\Rightarrow \quad \left[1 + \tfrac{b}{N}x\partial_x)\right]^N f(x) \;\approx\; f\big((1+\tfrac{b}{N})^N x\big).$$

$$\Rightarrow \quad \exp(bx\partial_x)f(x) \;=\; \lim_{N\to\infty} f\big((1+\tfrac{b}{N})^N x\big) \;=\; f\Big(\underbrace{\big[\lim_{N\to\infty}(1+\tfrac{b}{N})^N\big]}_{=e^b} x\Big)$$

Hence we arrive at

$$\boxed{\exp(bx\partial_x)f(x) \;=\; f(e^b x)}$$

This is a scale transformation by the factor e^b, like the zoom of a camera.

Correction advice: There are several ways to prove this. 1P for any kind of meaningful proof, subtleties like higher orders and commuting limits are not required. 1P for the correct interpretation.

(b) The solution of the differential equation $x\partial_x\phi(x) = \lambda\phi(x)$ is a simple power function

$$\phi(x) = Ax^\lambda.$$

where $\lambda \in \mathbb{R}$ can take any value and where A is an overall proportionality constant (makes sense since eigenvectors are defined only up to their length). If we apply the exponential operator to this function we get

$$\exp(bx\partial_x)\phi(x) \;=\; \exp(b\lambda)\phi(x) \;=\; Ae^{b\lambda}x^\lambda \;=\; A(e^b x)^\lambda.$$

Remark: This exercise demonstrates that power laws are scale-free, that is, they are eigenfunctions (i.d. some kind of invariant) under rescaling.

$(\Sigma = 12P)$