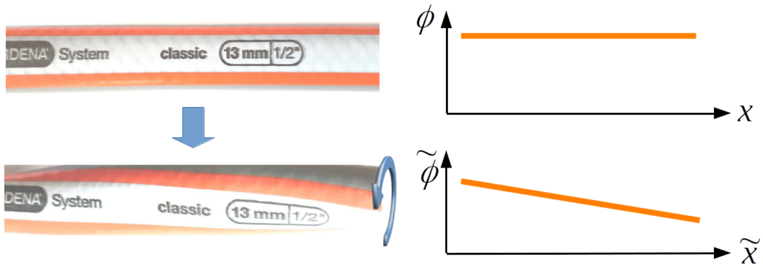
Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2021



Electrodynamics is the story of twisting a garden hose.

Exercise 7.1: Minimal coupling and covariant derivative (6P)

In this exercise we consider a non-relativistic classical and quantum particle in a given time-independent electromagnetic field $\phi(\mathbf{q}), \vec{A}(\mathbf{q})$, where $\mathbf{q} = \vec{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$. The aim is to illustrate the concept of *minimal coupling*. Starting point is the Lagrange function

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{q}}^2 - e\phi(\mathbf{q}) + e\dot{\mathbf{q}} \cdot \vec{A}(\mathbf{q})\,.$$

(a) Compute the generalized momentum $\mathbf{p}$. (1P)

(b) Verify that the Lagrange function is correct by determining the equations of motion and confirming that they are equivalent to the Lorentz force $\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$, where e is the electric charge, $\vec{v} = \dot{\mathbf{q}}$, $\vec{E} = -\mathrm{grad}(\phi) - \frac{\partial \vec{A}}{\partial t}$, and $\vec{B} = \mathrm{rot}(\vec{A})$. (2P)

(c) Compute the Hamilton function $H(\mathbf{q}, \mathbf{p})$ by the usual Legendre transformation and show that it is given by (2P)

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2m}\Big(\mathbf{p} - e\vec{A}(\mathbf{q})\Big)^2 + e\phi(\mathbf{q})$$

Remark: This boils down to a universal "cooking recipe" how to couple a system to an electromagnetic field which is known in the literature as *minimal coupling*: Simply add the potential $e\phi$ and replace the generalized momentum $\vec{p} \to \vec{p} - e\vec{A}$.

(d) Finally consider the quantum case and show that the Hamiltonian is given by

$$\mathbf{H} = -\frac{\hbar^2}{2m} \sum_{j=1}^{3} \mathbf{D}_j^2 + e\phi(\mathbf{q})$$

where $\mathbf{D}_j = \partial_j - \frac{ie}{\hbar} A_j$ is the so-called *covariant derivative*. (1P)

Exercise 7.2: Kaluza-Klein theory (6P)

Standard (quantum) electrodynamics regards the $U(1)$ circle as a separate additional structure attached to space-time. A very interesting alternative approach is the so-called Kaluza-Klein theory, which interprets the $U(1)$ circle as an additional fifth dimension, integrating it into space-time and applying general relativity (see lecture notes). With this exercise we want to get into touch with some aspects of the *Kaluza-Klein miracle*.

Please turn over ⇒

(a) Consider a garden hose with coordinate x and a compactified coordinate $\phi \in [0, 2\pi]$. Let us for simplicity ignore the periodicity, treating ϕ as if $\phi \in \mathbb{R}$ was unrestricted. Suppose that without a twist, the metric tensor on the hose is given by

$$g_{ij} = \begin{pmatrix} g_{xx} & g_{x\phi} \\ g_{\phi x} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & \\ & \rho^2 \end{pmatrix},$$

where ρ is the radius of the tube. Now let us twist the garden hose uniformly, as shown in the figure above. This will correspond to a coordinate transformation $\tilde{x} = x$, $\tilde{\phi} = \phi - ax$, where $-a$ is the slope of the orange line in the figure. Compute the metric tensor $\tilde{g}_{ij}$ in the coordinates $\tilde{x}, \tilde{\phi}$. (2P)

(b) Repeat part (a), but now applied to a 1+3+1-dimensional garden hose with coordinates $(x^0, x^1, x^2, x^3, x^4) = (ct, x, y, z, \phi)$ and the metric $g_{ij} = \mathrm{diag}(-1, 1, 1, 1, \rho^2)$. Let us use Latin indices for the range 0...4 and Greek indices for the range 0...3. Suppose that the garden hose is deformed uniformly by $\tilde{x}^\mu = x^\mu$, $\tilde{x}^4 = x^4 - \frac{e}{\hbar} A_\mu x^\mu$, where e is the elementary charge and A^μ is the electromagnetic 4-potential which is assumed here to be constant. Compute $\tilde{g}_{ij}$ in the coordinates $\tilde{x}^i$. (1P)

(c) Calculate the determinant $\tilde{g} = \det(\tilde{g}_{ij})$ and show that the inverse $\tilde{g}^{ij}$ in block notation is given by (1P)

$$\tilde{g}^{ij} = \left(\begin{array}{c|c} \eta^{\mu\nu} & -\frac{e}{\hbar} A^\mu \\ \hline -\frac{e}{\hbar} A^\nu & \frac{e^2}{\hbar^2} A_\sigma A^\sigma + \rho^{-2} \end{array} \right).$$

(d) A central idea of the Kaluza-Klein theory is that motion along the extra dimension is nothing but electric charge. Quantum-mechanically, as the extra dimension $\phi = x^4$ is compactified modulo 2π, this motion is quantized, explaining nicely the observed quantization of electric charge. Let us play a little bit with these thoughts by considering a factorized wave function

$$\Psi(x^0, \ldots, x^4) = \psi(x^0, \ldots, x^3) e^{in\phi},$$

where $n \in \mathbb{Z}$ counts the number of elementary charges. In the following we would like to solve the massless wave equation $\Box\Psi = 0$ in the geometry given by $\tilde{g}$. However, since $\tilde{g}^{ij}$ is non-diagonal, we have to replace $\Box = \partial_i \partial^i$ by the so-called *Laplace-Beltrami operator* (cf. previous exercise)

$$\Box_{\tilde{g}} = \frac{1}{\sqrt{|\tilde{g}|}} \partial_i \sqrt{|\tilde{g}|} \tilde{g}^{ij} \partial_j .$$

Write out the wave equation $\Box_{\tilde{g}} \Psi = 0$ in components. As you will see, the Kaluza-Klein theory is consistent with the principle of minimal coupling. (2P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, June 04 at 12:00.