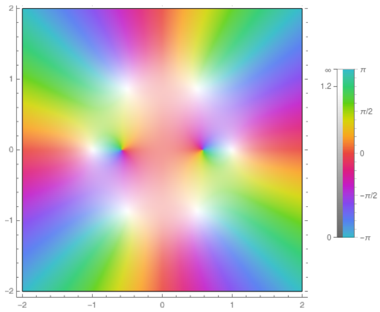

Complex fields can be viewed with phases represented as colors (Mathematica: `ComplexPlot[...]`)

EXERCISE 6.1: COMPLEX SCALAR FIELD (4P)

Let us consider the following action for a pair of real-valued scalar fields $\psi(\mathbf{x})$ and $\chi(\mathbf{x})$:

$$S[\psi,\chi] \;=\; -\frac{1}{2}\int\Big(\eta^{\mu\nu}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x^\nu}+\eta^{\mu\nu}\frac{\partial\chi}{\partial x^\mu}\frac{\partial\chi}{\partial x^\nu}\Big)\,\mathrm{d}^4x$$

(a) Determine the equations of motion. (1P)

(b) Rewrite the action as $S[\phi,\phi^*]$ in terms of a complex-valued field $\phi=\frac{1}{\sqrt{2}}(\chi+i\psi)$ and $\phi^*=\frac{1}{\sqrt{2}}(\chi-i\psi)$. (1P)

(c) Find the equations of motion for $\phi(\mathbf{x})$ and $\phi^*(\mathbf{x})$ (treating ϕ and ϕ^* as independent) and confirm that they are compatible with (a). (1P)

(d) Extend the action $S[\phi,\phi^*]$ by a mass term and derive the equations of motion. (1P)

EXERCISE 6.2: INVARIANTS OF THE ELECTROMAGNETIC FIELD TENSOR (8P)

(a) A Lorentz transformation is an isometry in Minkowski space. Show that this implies that the determinant of a Lorentz transformation matrix $\Lambda^\mu{}_\nu$ equals ±1. (2P)

(b) Demonstrate that the 4-dimensional fully antisymmetric Levi-Civita symbols $\epsilon_{\alpha\beta\gamma\delta}$ with $\epsilon^{\alpha\beta\gamma\delta}:=-\epsilon_{\alpha\beta\gamma\delta}$ are invariant under special Lorentz transformations $SO(1,3)$ while they may change their sign under general Lorentz transformations $O(1,3)$. Note: This is the reason why they are called *symbols* and not *tensor*. (3P)

(c) The dual field tensor of the electromagnetic field is defined as ${}^\star F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\tau}F^{\rho\tau}$. Compute ${}^\star F_{\mu\nu}$ as well as the three Lorentz-invariant quantities

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu}\,,\qquad \frac{1}{4}\,{}^\star F_{\mu\nu}F^{\mu\nu}\,,\qquad \frac{1}{2}\,{}^\star F_{\mu\nu}\,{}^\star F^{\mu\nu}$$

in terms of the fields $\vec{E}$ and $\vec{B}$. (2P)

(d) Is it possible to convert a pure electric field into a pure magnetic field by Lorentz transformation? (1P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, May 28 at 12:00.