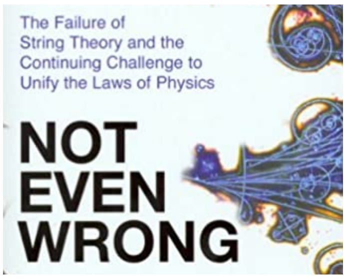
Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2021



Peter Woits slating review of String Theory.

Exercise 5.1: Polynomial action (3P)

In the lecture we derived the action of a point particle with mass m in a potential V:

$$S[\mathbf{x},\dot{\mathbf{x}}] = \int_{\tau_A}^{\tau_B} \Big(-mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + V(\mathbf{x})\Big)\,\mathrm{d}\tau.$$

Some theorists do not like this action because of the nasty square root. They would rather prefer a *polynomial* action. In this exercise let us study the following polynomial action

$$S[\mathbf{x},\dot{\mathbf{x}},\xi] = \int_{\tau_A}^{\tau_B} L(\mathbf{x},\dot{\mathbf{x}},\xi)\,\mathrm{d}\tau = \int_{\tau_A}^{\tau_B} \Big(\frac{1}{2\xi}\dot{x}_\mu\dot{x}^\mu - \frac{\xi m^2c^2}{2} + V(\mathbf{x})\Big)\,\mathrm{d}\tau,$$

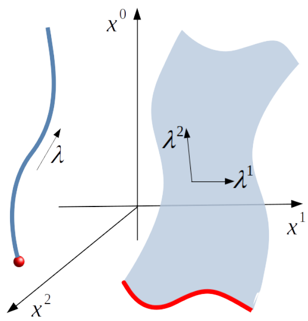
where $\xi(\tau)$ is an additional independent function.

(a) Compute the variation $\delta S = 0$ with respect to δx^μ and $\delta\xi$ and derive the corresponding equations of motion. (2P)

(b) Insert the solution for $\xi(\tau)$ into the other equation of motion and show that we get the same results as for the ordinary action (see lecture notes). (1P)

Exercise 5.2: String Theory (9P)

So far we have understood the physics of a relativistic particle. Such a particle is a zero-dimensional object which moves on a one-dimensional **world line** parameterized e.g. by $\lambda \mapsto \mathbf{x}(\lambda)$. For the action we have simply chosen the line integral over the relativistic length element $\mathrm{d}s = \sqrt{-\dot{x}_\mu\dot{x}^\mu}\,\mathrm{d}\lambda$.



Let us now do string theory. A string is a one-dimensional object which moves on a two-dimensional **world sheet** parameterized two parameters, e.g. by $\lambda^1,\lambda^2 \mapsto \mathbf{x}(\lambda^1,\lambda^2)$.

(a) As usual, the relativistic line element in the embedding space $\mathbb{R}_{3+1}$ is $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. On the world sheet the corresponding line element can be expressed as $\mathrm{d}s^2 = g_{ij}\,\mathrm{d}\lambda^i\,\mathrm{d}\lambda^j$, where the Latin indices run over 1,2 and where the metric g_{ij} depends on the position on the world sheet. Give a general expression for g_{ij}. (1P)

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(b) Prove: In the ordinary Euclidean $\mathbb{R}^n$ an infinitesimal area element $\mathrm{d}A$ spanned by two vectors $\mathrm{d}\vec{a}$ and $d\vec{b}$ is given by $\mathrm{d}A = \sqrt{(\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{a})(\mathrm{d}\vec{b}\cdot\mathrm{d}\vec{b}) - (\mathrm{d}\vec{a}\cdot\mathrm{d}\vec{b})^2}$. (1P)

(c) Guess an analogous formula for the area element spanned by $\mathrm{d}\mathbf{a}$ and $\mathrm{d}\mathbf{b}$ in the 3+1-dimensional Minkowski space. (1P)

(d) Let $\mathrm{d}\mathbf{a}$ and $\mathrm{d}\mathbf{b}$ be the displacements on the world sheet due to a variation of the parameters by $\mathrm{d}\lambda^1$ and $\mathrm{d}\lambda^2$, respectively. Show that the area element is given by $\mathrm{d}A = \sqrt{|g|}\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2$, where g is the determinant of g_{ij}. (1P)

(e) The **Nambu-Goto action** is defined as being proportional to the area of the sheet

$$S_{NG} = -T\int \mathrm{d}A$$

where $T > 0$ is a coupling constant (string tension). In a given parameterization of the world sheet, this action can be expressed as

$$S_{NG} = \int \mathcal{L}_{NG}\Big(\{\tfrac{\partial x^\mu}{\partial\lambda^1}\},\{\tfrac{\partial x^\nu}{\partial\lambda^2}\}\Big)\,\mathrm{d}\lambda^1\,\mathrm{d}\lambda^2.$$

Determine the Lagrange density $\mathcal{L}_{NG}$. (1P)

(f) Analogous to the particle momentum $p_\mu = \frac{\partial L}{\partial\dot{x}^\mu}$, the string momentum is defined as

$$\Pi^i_\mu = \frac{\partial\mathcal{L}_{NG}}{\partial\left(\frac{\partial x^\mu}{\partial\lambda^i}\right)}.$$

As there is no potential, the equations of motion are then simply given by $\partial_i\Pi^i_\mu = 0$. Show that these classical equations of motion are equivalent to the analog of the wave equation $\Box x^\mu = 0$, where $\Box$ is the so-called **Laplace-Beltrami operator**: (2P)

$$\Box x^\mu = 0\,,\qquad \Box = \frac{1}{\sqrt{|g|}}\partial_i\sqrt{|g|}\,g^{ij}\partial_j$$

Hint: g^{ij} is the inverse matrix of g_{ij}.

(g) String theorist often prefer a different action, namely, the **Polyakov action**

$$S_P = -\frac{1}{4\pi\alpha}\int \mathrm{d}\lambda^1\,\mathrm{d}\lambda^2\sqrt{|g|}\,g^{ij}\,\partial_i x^\mu\partial_j x^\nu\eta_{\mu\nu}.$$

This action is sometimes easier to handle since it is polynomial in the coordinates (no square root). Show that this action renders exactly the same equations of motion. (2P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, May 21 at 12:00.