SAMPLE SOLUTIONS EXERCISE 1

EXERCISE 1.1: POLYNOMIAL VECTOR SPACE, BASIS AND DUAL BASIS (6P)

The purpose of this exercise is to see that vectors can be realized in various forms. Here we consider the example of second-order polynomials.

(a) Consider the set of 2nd-order polynomials

$$V = \left\{ p \,\middle|\, p(x) = a_0 + a_1 x + a_2 x^2 \right\}$$

Show that this set equipped with the operations

$$\begin{aligned} '+' : \quad & (p+q)_{(x)} := p(x) + q(x) \qquad && p, q \in V \\ ' \cdot ' : \quad & \big(\lambda p\big)_{(x)} := \lambda\, p(x) \qquad && p \in V, \lambda \in \mathbb{R} \end{aligned}$$

is a vector space over $\mathbb{R}$. (1P)

(b) Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defined by $\mathbf{e}_i(x) = x^{i-1}$ is a basis of V. (1P)

(c) The dual vector space V^* consist of 1-forms that map polynomials to real numbers. Therefore, the operation β^x of evaluating a polynomial at the position x is obviously an element of V^*. More specifically, for all $p \in V$ and $x \in \mathbb{R}$ let us consider

$$\beta^x \in V^* : \quad \beta^x(p) := p(x)\,.$$

Prove that $\{\beta^1, \beta^2, \beta^3\}$ are linearly independent, providing a basis of V^*. (1P)

(d) The *dual basis* $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ is defined by the fundamental relation $\mathbf{e}^j(\mathbf{e}_i) = \delta_i^j$. Express the dual basis vectors as linear combinations of the 1-forms $\{\beta^1, \beta^2, \beta^3\}$. As usual, you may use *Mathematica*® or similar tools. (2P)

(e) Represent the 1-form

$$\gamma \in V^* : V \mapsto \mathbb{R} : \qquad \gamma(p) := \int_0^1 p(x)\, \mathrm{d}x$$

as a linear combination $\gamma = \mu_k \mathbf{e}^k$ in the dual basis. (1P)

Remark: You can verify your results in (d) and (e) by proving that

$$\gamma(p) = \frac{23}{12} p(1) - \frac{4}{3} p(2) + \frac{5}{12} p(3)\,.$$

As you can see, this functional, as strange as it looks like, correctly integrates second-order polynomials in the range from 0 to 1.

SAMPLE SOLUTION

(a) Verify the vector space axioms (trivial):

- V is a commutative group under addition.
- Identity element of scalar multiplication.
- Compatibility laws (distributive laws).

  Correction advice: This part is correctly solved if it is pointed out that the vector space axioms are satisfied since addition and multiplication is defined as addition and multiplication of numbers, satisfying the same distributive laws.

(b) We have $\mathbf{e}_1(x) = 1$, $\mathbf{e}_2(x) = x$, $\mathbf{e}_3(x) = x^2$. First we have to show that these vectors are linearly independent:

$$\mu^i \mathbf{e}_i = 0 \quad \Leftrightarrow \quad \mu^i \mathbf{e}_i(x) = 0 \,\forall x \quad \Leftrightarrow \quad \mu^1 = \mu^2 = \mu^3 = 0\,.$$

Secondly we have to show that every other vector can be represented as a linear combination of these vectors. Take any $p \in V$ with $p(x) = a_0 + a_1 x + a_2 x^2$. Then it is clear that $p = \mu^i \mathbf{e}_i$ with $\mu^i \equiv a_{i-1}$. Therefore, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a valid basis.

(c) The forms $\{\beta^1, \beta^2, \beta^3\}$ applied to $p(x) = a_0 + a_1 x + a_2 x^2$ give the result

$$\begin{aligned} \beta^1(p) &= a_0 + a_1 + a_2 \\ \beta^2(p) &= a_0 + 2a_1 + 4a_2 \\ \beta^3(p) &= a_0 + 3a_1 + 9a_2 \end{aligned}$$

In order to prove that they are also linearly independent we have to show that any vanishing linear combination of them implies that the corresponding coefficients are zero. In fact, let us assume that $\nu_i \beta^i = 0$, that is

$$\nu_i \beta^i(p) \;=\; (\nu_1 + \nu_2 + \nu_3) a_0 + (\nu_1 + 2\nu_2 + 3\nu_3) a_1 + (\nu_1 + 4\nu_2 + 9\nu_3) a_2 = 0$$

for all p, or equivalently, for all $\{a_0, a_1, a_2\}$. This means that the brackets have to vanish, leading to a system of three linear equations

$$\begin{aligned} \nu_1 + \nu_2 + \nu_3 &= 0 \\ \nu_1 + 2\nu_2 + 3\nu_3 &= 0 \\ \nu_1 + 4\nu_2 + 9\nu_3 &= 0 \end{aligned}$$

which has the only solution $\nu_1 = \nu_2 = \nu_3 = 0$. This implies that the forms $\{\beta^1, \beta^2, \beta^3\}$ are indeed linearly independent. Since V is 3-dimensional, V^* is also 3-dimensional (see lecture notes), hence $\{\beta^1, \beta^2, \beta^3\}$ are a basis of V^*.

(d) We want to express the $\mathbf{e}^j$ as linear combinations of the β^k, that is, we want to determine coefficients ${C^j}_k$ such that

$$\mathbf{e}^j \;=\; {C^j}_k \beta^k.$$

In order to determine these coefficients, we apply both sides of the equation to the basis vector $\mathbf{e}_i$, giving (1P)

$$\delta^j_i \;=\; \mathbf{e}^j(\mathbf{e}_i) \;=\; {C^j}_k \beta^k(\mathbf{e}_i)\,, \qquad\qquad (*)$$

where $\beta^k(\mathbf{e}_i)$ can be evaluated by

$$\beta^k(\mathbf{e}_1) = 1\,, \quad \beta^k(\mathbf{e}_2) = k\,, \quad \beta^k(\mathbf{e}_3) = k^2 \qquad \Leftrightarrow \qquad \beta^k(\mathbf{e}_i) = k^{i-1}\,.$$

Therefore, the system of equations (*) can be written in matrix form as

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} {C^1}_1 & {C^1}_2 & {C^1}_3 \\ {C^2}_1 & {C^2}_2 & {C^2}_3 \\ {C^3}_1 & {C^3}_2 & {C^3}_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

Thus, all what we have to do is to invert the matrix e.g. with *Mathematica*®
(1P)

$$\begin{pmatrix} {C^1}_1 & {C^1}_2 & {C^1}_3 \\ {C^2}_1 & {C^2}_2 & {C^2}_3 \\ {C^3}_1 & {C^3}_2 & {C^3}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -\frac{5}{2} & 4 & -\frac{3}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}.$$

Remark: Optionally, we can write out the result as

$$\begin{aligned} \mathbf{e}^1(p) &= 3p(1) - 3p(2) + p(3) \\ \mathbf{e}^2(p) &= -\frac{5}{2}p(1) + 4p(2) - \frac{3}{2}p(3) \\ \mathbf{e}^3(p) &= \frac{1}{2}p(1) - p(2) + \frac{1}{2}p(3) \end{aligned}$$

which holds for all polynomials p. Now one can easily verify the result by inserting the basis polynomials $\mathbf{e}_1(x) = 1$, $\mathbf{e}_2(x) = x$, $\mathbf{e}_3(x) = x^2$ and checking that $\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j$.

(e) The 1-form $\gamma \in V^*$ can be represented as $\gamma = \mu_k \mathbf{e}^k$ with components $\mu_k = \gamma(\mathbf{e}_k)$ (see lecture notes). These components can be evaluated easily as follows:

$$\mu_k = \gamma(\mathbf{e}_k) = \int_0^1 x^{k-1}\,\mathrm{d}k = \frac{x^k}{k}\Big|_0^1 = \frac{1}{k}$$

hence

$$\gamma = \mathbf{e}^1 + \frac{1}{2}\mathbf{e}^2 + \frac{1}{3}\mathbf{e}^3.$$

Remark: Inserting the results of (d) we get the strange result

$$\gamma(p) = \frac{23}{12}p(1) - \frac{4}{3}p(2) + \frac{5}{12}p(3).$$

Nevertheless this result is correct. To see this insert a general 2nd order polynomial $p(x) = ax^2 + bx + c$, giving $\gamma(p) = \frac{a}{3} + \frac{b}{2} + c$, and this equals the integral from 0 to 1 over this polynomial. Of course, this identity is only valid for 2nd order polynomials on which we restricted here in this exercise. What you see here is that in such a 3D space every linear functional in $V*$ is uniquely given if we know its behavior at three different positions (here $x = 1, 2, 3$).

(Σ = 6P)