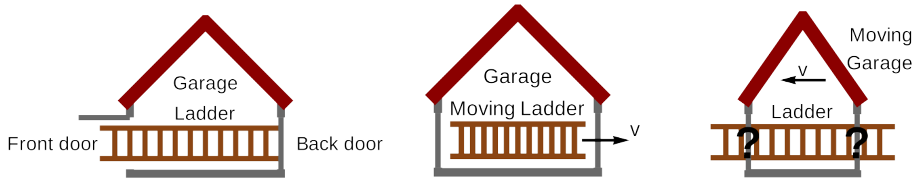

[Ladder Paradox (Wikimedia)]

Exercise 3.1: Ladder Paradox (6P)

A physicist is faced with the problem that his/her ladder does not fit into the garage because the rest length of the ladder exceeds the size of the garage by 25% (see figure). But apparently there is a simple relativistic solution: We would move the ladder into the garage at 80% of the velocity of light, and then we would instantly close the doors just in the moment when the Lorentz-contracted ladder fits into the garage. However, from the perspective of the ladder, the garage is Lorentz-contracted, so it seems to be impossible to close the doors.

(a) Let a be the size of the garage in its rest frame. Compute β and γ. Is the Lorentz-contracted length b of the ladder in fact smaller than the size of the garage? (1P)

(b) Let us first keep both doors open and let the ladder pass freely. Draw the Minkowski diagrams in the frames of the garage and the ladder in exact proportions (maßstabsgetreu). (2P)
Analyze the sequence of the following four events:
A: Tip of ladder enters garage
B: End of ladder enters garage
C: Tip of ladder leaves garage
D: End of the ladder leaves garage. (1P)

(c) The physicist in the rest frame of the garage says: “In the right moment I can close both doors”. An observer co-moving with the ladder says: “The ladder is too long, therefore it is in principle impossible to close both doors, even at different times”. Who is right, who is wrong, and why? (2P)

Exercise 3.2: Distance-dependent solution of the wave equation (3P)

In this exercise we want to outline an interesting way finding solutions of the wave equations which depend exclusively on the relativistic distance $\mathbf{x}^2 = x_\mu x^\mu$. These solutions are then invariant under Lorentz transformations.

(a) Consider the $d+1$-dimensional Minkowski space equipped with the metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$. Let $s = |\mathbf{x}| = \sqrt{x_\mu x^\mu}$ be the relativistic distance. Show that for a function $g(s)$ which depends only on s, the partial differential equation

$$\Box g = \partial_\mu \partial^\mu g(s) = 0 \qquad\qquad s \neq 0$$

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reduces to an autonomous[1] ordinary differential equation for $g(s)$ in terms of s. To find this ODE, consider first the 1+1-dimensional case, then the 2+1-dimensional case and finally guess the form of the ODE for general $d \in \mathbb{N}$. (2P)

(b) Solve the ODE for general d obtained in (a). Note the special case $d = 1$. (1P)

Exercise 3.3: Fourier transformations in Minkowski space (3P)

In the $(d+1)$-dimensional Minkowski space, the Fourier transformation $f(\mathbf{x}) \mapsto \tilde{f}(\mathbf{k})$ is defined by

$$\tilde{f}(\mathbf{k}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathrm{d}^{d+1}x \; e^{ik_\mu x^\mu} \, f(\mathbf{x}) \,.$$

(a) Write down the inverse Fourier transformation. (1P)

(b) The Greens function of the Klein-Gordon equation is defined by

$$(\Box - m^2) G(\mathbf{x}) = \delta^{d+1}(\mathbf{x}).$$

Calculate the Fourier transformation of this equation and find a solution for the transformed Greens function $\tilde{G}(\mathbf{k})$. (1P)

(c) Prove the following statement: If a given function $f(\mathbf{x})$ is Lorentz-invariant under $\mathbf{x} \to \Lambda\mathbf{x}$, then the Fourier-transformed function $\tilde{f}(\mathbf{k})$ will also be Lorentz-invariant under $\mathbf{k} \to \Lambda\mathbf{k}$. (1P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, May 07 at 12:00.

[1] Autonomous means that only the variable s and derivatives with respect to s occur, while the coordinates x^μ do no longer appear individually.