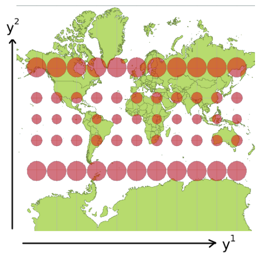
SPECIAL RELATIVITY AND CLASSICAL FIELD THEORY

LECTURE AND TUTORIAL – PROF. DR. HAYE HINRICHSEN – MAXIMILIAN ZEMSCH – SS 2021

Mercator projection [Wikimedia]

EXERCISE 2.1: MERCATOR PROJECTION (3P)



Nautical maps use the Mercator projection. The Mercator projection maps the coordinates of the sphere (the longitude $x^1 = \phi \in [0, 2\pi]$ and the latitude $x^2 = \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ measured from the equator) onto new coordinates y^1, y^2 on the map. The longitude is mapped identically by $y^1 \equiv x^1$, while the latitude is stretched by a certain function $y^2 := f(x^2)$. This function is chosen in such a way that the map preserves angles and proportions, but not the area (see figure). These properties manifests themselves in the fact that the matrix of the metric tensor on the map is proportional to the identity.

(a) Construct the metric tensor $g_{\mu\nu}(\phi, \theta) = \begin{pmatrix} g_{\phi\phi} & g_{\phi\theta} \\ g_{\theta\phi} & g_{\theta\theta} \end{pmatrix}$ on the sphere. (1P)

(b) Compute the metric tensor $\tilde{g}_{ij}(y^1, y^2)$ on the map for general f. (1P)

(c) Determine the function f in such a way that the map is angular-preserving. (1P)

EXERCISE 2.2: SIMPLE LIE ALGEBRAS AND THE EXPONENTIAL FUNCTION (6P)

Consider an abstract operator λ obeying $\lambda^2 = -\mathbb{1}$, where $\mathbb{1} = \lambda^0$ is the identity.

(a) Write down the Taylor series of $\Lambda = \exp(\phi\lambda)$, where $\phi \in \mathbb{R}$. (1P)

(b) Separate the Taylor series into an even and an odd part in order to show that (2P)

$$\exp(\phi\lambda) = \mathbb{1}\cos\phi + \lambda\sin\phi.$$

(c) Apply the result from (b) to the representations $\lambda = i$ and $\lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. (1P)

(d) Repeat (b) for $\lambda^2 = +\mathbb{1}$ and find a non-trivial 2×2 matrix representation. (2P)

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Exercise 2.3: Operator exponential function **(3P)**

In the last lecture we studied the translation operator $\exp(a\partial_x)$. Let us now consider the operator $\exp(bx\partial_x)$ acting on functions $f : \mathbb{R} \to \mathbb{R}$ with a real parameter $b > 0$.

(a) Compute $\exp(bx\partial_x)f(x)$. (*Hint:* Consider $\exp(bx\partial_x) = [\exp(\frac{b}{N}x\partial x)]^N$ for $N \to \infty$). What kind of transformation does this operator describe? (2P)

(b) Determine the eigenfunctions and eigenvalues of the generator $x\partial_x$, that is, find functions $\phi : \mathbb{R} \to \mathbb{R}$ and eigenvalues $\lambda \in \mathbb{R}$ such that $x\partial_x\phi(x) = \lambda\phi(x)$. Use the result to compute $\exp(bx\partial_x)\phi(x)$. (1P)

(Σ = 12P)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, April 30 at 12:00.