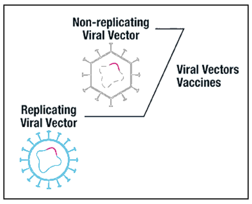
# Special Relativity and Classical Field Theory

Lecture and Tutorial – Prof. Dr. Haye Hinrichsen – Maximilian Zemsch – SS 2021




Vectors everywhere [Wikimedia]

## Exercise 1.1: Polynomial vector space, basis and dual basis (6P)

The purpose of this exercise is to see that vectors can be realized in various forms. Here we consider the example of second-order polynomials.

(a) Consider the set of 2nd-order polynomials

$$V = \left\{ p \,\middle|\, p(x) = a_0 + a_1 x + a_2 x^2 \right\}$$

Show that this set equipped with the operations

$$\begin{aligned} '+' : \quad & (p+q)_{(x)} := p(x) + q(x) \qquad && p, q \in V \\ ' \cdot ' : \quad & (\lambda p)_{(x)} := \lambda\, p(x) && p \in V, \lambda \in \mathbb{R} \end{aligned}$$

is a vector space over $\mathbb{R}$. (1P)

(b) Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ defined by $\mathbf{e}_i(x) = x^{i-1}$ is a basis of $V$. (1P)

(c) The dual vector space $V^*$ consist of 1-forms that map polynomials to real numbers. Therefore, the operation $\beta^x$ of evaluating a polynomial at the position $x$ is obviously an element of $V^*$. More specifically, for all $p \in V$ and $x \in \mathbb{R}$ let us consider

$$\beta^x \in V^* : \quad \beta^x(p) := p(x)\,.$$

Prove that $\{\beta^1, \beta^2, \beta^3\}$ are linearly independent, providing a basis of $V^*$. (1P)

(d) The *dual basis* $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ is defined by the fundamental relation $\mathbf{e}^j(\mathbf{e}_i) = \delta_i^j$. Express the dual basis vectors as linear combinations of the 1-forms $\{\beta^1, \beta^2, \beta^3\}$. As usual, you may use *Mathematica*® or similar tools. (2P)

(e) Represent the 1-form

$$\gamma \in V^* : V \mapsto \mathbb{R} : \qquad \gamma(p) := \int_0^1 p(x)\, \mathrm{d}x$$

as a linear combination $\gamma = \mu_k \mathbf{e}^k$ in the dual basis. (1P)

**Remark:** You can verify your results in (d) and (e) by proving that

$$\gamma(p) = \frac{23}{12} p(1) - \frac{4}{3} p(2) + \frac{5}{12} p(3)\,.$$

As you can see, this functional, as strange as it looks like, correctly integrates second-order polynomials in the range from 0 to 1.

($\Sigma$ = **6P**)

Please submit your solution as a single pdf file via WueCampus. Deadline is Friday, April 23 at 12:00. Normally we have 12P per sheet. This week we have a warmup exercise with only 6P.